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PRIMAL-DUAL AND GENERAL PRIMAL-DUAL PARTITIONS IN LINEAR SEMI-INFINITE PROGRAMMING WITH BOUNDED COEFFICIENTS

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Abstract

We consider two partitions over the space of linear semi-infinite programming parameters with a fixed index set and bounded coefficients (the functions of the constraints are bounded). The first one is the primal-dual partition inspired by consistency and boundedness of the optimal value of the linear semi-infinite optimization problems. The second one is a refinement of the primal-dual partition that arises considering the boundedness of the optimal set. These two partitions have been studied in the continuous case, this is, the set of indices is a compact infinite compact Hausdorff topological space and the functions defining the constraints are continuous. In this work, we present an extension of this case. We study same topological properties of the cells generated by the primal-dual partitions and characterize their interior. Through examples, we show that the results characterizing the sets of the partitions in the continuous case are neither necessary nor sufficient in both refinements. In addition, a sufficient condition for the boundedness of the optimal set of the dual problem has been presented.

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Introduction

We associate with each triplet $\pi \in \Pi = B^n \times \mathrm{B} \times \mathbb{R}$ a primal problem

$$P: \quad \inf \boldsymbol{c}'\boldsymbol{x}$$
$$s.t. \quad \boldsymbol{a}_t\boldsymbol{x} \geq b_t, \quad t \in T$$

and a dual problem in the sense of Haar

$$D: \quad \sup \sum_{t \in T} \lambda_t b_t$$
$$s.t. \sum_{t \in T} \lambda_t \boldsymbol{a}_t = \boldsymbol{c}$$
$$\lambda \in \mathbb{R}_+^{(T)}.$$

Above and henceforth, $\mathbb{R}_+^{(T)}$ denotes the set of nonnegative general finite sequences, that is, functions $\lambda: T \to \mathbb{R}_+$ satisfying that $\lambda_t = 0$ for all $t \in T$ except maybe for a finite number of indices. In $\mathbb{R}_+^{(T)}$ we consider the norms l_∞ and l_1.

As both primal and dual problems are defined with the same data $\boldsymbol{a}$, b and $\boldsymbol{c}$, these are represented by the triplet $\pi := (\boldsymbol{a}, b, \boldsymbol{c})$. The parameters space Π is defined as the set of all triplets π with n and T fixed, equipped with the pseudometrics $d : \Pi \times \Pi \to [0, \infty]$, defined by

$$d(\pi^1, \pi^2) := \max\left\{ \|\boldsymbol{c}^1 - \boldsymbol{c}^2\|_\infty, \sup_{t \in T} \left\| \begin{pmatrix} \boldsymbol{a}_t^1 \\ b_t^1 \end{pmatrix} - \begin{pmatrix} \boldsymbol{a}_t^2 \\ b_t^2 \end{pmatrix} \right\|_\infty \right\}$$

where, $\pi^i = (\boldsymbol{a}^i, b^i, \boldsymbol{c}^i) \in \Pi$, $i = 1, 2$ and $\|\cdot\|_\infty$ represents the uniform norm.

The partitions where T is an infinite compact Hausdorff topological space and the functions $\boldsymbol{a}$ and b are continuous, have been analyzed in [1], [5] and [6]. In particular, the last reference deals with the consistence and boundedness of the optimal value of the problems and these properties define the primal-dual partition. The interior of the sets generated through the partition is also studied. In [8], partitions corresponding to an arbitrary index set T and arbitrary functions $\boldsymbol{a}$ and b are considered. In the present paper a refinement of the primal-dual partition is presented. Trough the article we shall consider only bounded linear semi-infinite optimization problems. The new sets of the partition arise from the boundedness of the optimal set of the optimization problems. This work extends the study of the primal-dual partition and its refinement to the case of bounded coefficients.

The new results are presented in Sections 3 and 4. In Section 3, we characterize the interior of the sets that are generated by the primal-dual partition and we show that the characterization is like that one obtained in the continuous case. In Section 4 we show that the conditions that characterize the sets generated by the refinement are neither necessary nor sufficient. In addition, we present a condition that implies the boundedness of the optimal set of the dual problem.

2. PRELIMINARY

This section begins with the notations to follow in the rest of the work. We denote by $\mathbb{R}_+$ the set of positive real numbers, and by $\mathbb{R}_{++}$ the set of positive real numbers where the zero is not included. In the n-dimensional space $\mathbb{R}^n$ endowed with the Euclidean norm, $\boldsymbol{x}'$ stands for the transpose of the vector column $\boldsymbol{x}$, the null vector will be denoted by $\boldsymbol{0}_n$. If X is a set of any topological space, $int\ X$ and $cl\ X$

will denote the interior and closure, respectively. Given a nonempty set $X \subset \mathbb{R}^n$, $conv\ X$ and $cone\ X$ will denote its convex and conical hull, respectively.

If C is a nonempty convex set, its recession cone $O^+(C)$ is:

$$cone\ \{\boldsymbol{y} \in \mathbb{R}^n : \boldsymbol{x} + \alpha\boldsymbol{y} \in C \text{ for all } \boldsymbol{x} \in C \text{ and for all } \alpha > 0\}.$$

The feasible set (optimal) of P and D will be denoted by F (F^*) and Λ (Λ^*), respectively. The optimal value of the primal (dual) problem P (D) will be denoted by $v^P(\pi)$ ($v^D(\pi)$) where, as usual, $v^P(\pi) = \infty$ and $v^D(\pi) = -\infty$ when the corresponding problems become inconsistent.

With each parameter π we associate the first and second moment cones $M := cone\ \{\boldsymbol{a}_t,\ t \in T\}$ and $N := cone\ \{(\boldsymbol{a}_t, b_t)',\ t \in T\}$, and its characteristic cone $K := cone\ \{(\boldsymbol{a}_t, b_t)',\ t \in T;\ (\boldsymbol{0}_n, -1)'\}$. Remember that π satisfies the Slater condition if there exist $\overline{\boldsymbol{x}} \in \mathbb{R}^n$ such that, $a_t'\overline{\boldsymbol{x}} > b_t$ for all $t \in T$. Also, π satisfies the strong Slater condition if there are $\varepsilon > 0$ and $\overline{\boldsymbol{x}} \in \mathbb{R}^n$, such that, $a_t'\overline{\boldsymbol{x}} \geq b_t + \varepsilon$ for all $t \in T$. From the definition we have that every parameter that satisfies the strong Slater condition is consistent and satisfies the Slater condition. However, the opposite is not true in general (see Example 4). In [4, Theorem] it is shown that a parameter satisfies the strong Slater condition if and only if $\boldsymbol{0}_{n+1} := (\boldsymbol{0}_n, 0)' \notin cl\ G$, where $G := conv\ \{(\boldsymbol{a}_t, b_t)' : t \in T\}$. It is worth mentioning that in the continuous case strong Slater and Slater conditions coincide.

We will denote by Π_C^P, Π_{IC}^P, Π_B^P and Π_{UB}^P (Π_C^D, Π_{IC}^D, Π_B^D and Π_{UB}^D) the sets of parameters that have primal (dual) problem consistent, inconsistent, bounded (consistent with finite optimal value) and unbounded, respectively. Also, Π_S^P (Π_S^D) will denote the set of parameters with solvable primal (dual) problem which have bounded optimal set, while Π_N^P (Π_N^D) will denote the set of parameters with primal (dual) problem which is not solvable or has unbounded optimal set. In the continuous case, the sets Π_S^P and Π_S^D are characterized in [5]. These characterizations are presented in the next lemma.

Lemma 2.1.
(i) $\pi \in \Pi_S^P$ if and only if $(\boldsymbol{0}_n, 1)' \notin cl\ K$ and $\boldsymbol{c} \in int\ M$.
(ii) $\pi \in \Pi_S^D$ if and only if $\boldsymbol{c} \in M$ and π satisfies the Slater condition.

In the first primal-dual partition, presented in [6], the primal and dual problems are classified in inconsistent (IC), bounded (B) and unbounded (UB) classes. This partition is showed in the following table.

$(D)\backslash(P)$	IC	B	UB
IC	Π_4	Π_5	Π_2
B	Π_6	Π_1	
UB	Π_3		

Table 1

were,

$$\Pi_1 := \Pi_B^P \cap \Pi_B^D,\ \Pi_2 := \Pi_{UB}^P \cap \Pi_{IC}^D,\ \Pi_3 := \Pi_{IC}^P \cap \Pi_{UB}^D,\ \Pi_4 := \Pi_{IC}^P \cap \Pi_{IC}^D,\ \Pi_5 := \Pi_B^P \cap \Pi_{IC}^D \text{ and }$$
$$\Pi_6 := \Pi_{IC}^P \cap \Pi_B^D.$$

We conclude this section with the characterization of the sets Π_i, $i = 1, \dots, 6$ where M, N and K play a crucial role, look at [6]. The next theorem, proved in [8], holds for the general linear semi-infinite optimization, hence in the particular case when $\boldsymbol{a}$ and b are bounded, as well.

Theorem 2.2.
(i) $\pi \in \Pi_1$ if and only if $(\mathbf{0}_n, 1)' \notin cl\, N$ and $\boldsymbol{c} \in M$.
(ii) $\pi \in \Pi_2$ if and only if $(\mathbf{0}_n, 1)' \notin cl\, N$ and $(\{\boldsymbol{c}\} \times \mathbb{R}) \cap cl\, N = \emptyset$.
(iii) $\pi \in \Pi_3$ if and only if $\{\boldsymbol{c}\} \times \mathbb{R} \subseteq K$.
(iv) $\pi \in \Pi_4$ if and only if $(\mathbf{0}_n, 1)' \in cl\, N$ and $\boldsymbol{c} \notin M$.
(v) $\pi \in \Pi_5$ if and only if $\boldsymbol{c} \notin M$, $(\mathbf{0}_n, 1)' \notin cl\, N$ and $(\{\boldsymbol{c}\} \times \mathbb{R}) \cap cl\, N \neq \emptyset$.
(vi) $\pi \in \Pi_6$ if and only if $(\mathbf{0}_n, 1)' \in cl\, N$, $\boldsymbol{c} \in M$ and $\{\boldsymbol{c}\} \times \mathbb{R} \nsubseteq K$.

3. Primal-dual stability

In [6], the following theorem presents the characterization of the interior of the sets generated by the primal-dual partition in the continuous case. Only continuous perturbations are considered.

Theorem 3.1. Let $\pi \in C(T)^n \times C(T) \times \mathbb{R}$ a parameter in continuous case. The following assertions are hold:
(i) $\pi \in int\, \Pi_1$ if and only if π satisfies the Slater condition and $\boldsymbol{c} \in int\, M$.
(ii) $\pi \in int\, \Pi_2$ if and only if there exists $\boldsymbol{y} \in \mathbb{R}^n$ such that

$$\boldsymbol{c}'\boldsymbol{y} < 0 \text{ and } \boldsymbol{a}_t'\boldsymbol{y} > 0 \text{ for all } t \in T.$$

(iii) $\pi \in int\, \Pi_3$ if and only if $(\mathbf{0}_n, 1)' \in int\, N$.
(iv) $int\, \Pi_i = \emptyset$ for $i = 4, 5, 6$.

The theorem, obtained in [8], shows the results which characterize the interior of the sets generated by the primal-dual partition in the general case. Arbitrary perturbations are considered.

Theorem 3.2. Let $\pi \in \Pi$ a parameter. The following assertions hold:
(i) $\pi \in int\, \Pi_1$ if and only if π satisfies the strong Slater condition and $\boldsymbol{c} \in int\, M$.
(ii) $\pi \in int\, \Pi_2$ if and only if there exists $\boldsymbol{y} \in \mathbb{R}^n$ and $\delta > 0$ such that

$$\boldsymbol{c}'\boldsymbol{y} < 0 \text{ and } \boldsymbol{a}_t' y \geq 0 \text{ for all } t \in T.$$

(iii) *(a)* $\pi \in int\, \Pi_3$, $M = \mathbb{R}^n$ if and only if $(\mathbf{0}_n, 1)' \in int\, N$.
(b) $\pi \in int\, \Pi_3$, $M \neq \mathbb{R}^n$ if and only if

$$\mathbf{0}_n \notin int\, conv\, \{\boldsymbol{a}_t : t \in T\},\ (\mathbf{0}_n, 1)' \in O^+(cl\, G) \text{ and } \boldsymbol{c} \in int\, M.$$

(iv) $\pi \in int\, \Pi_4$ if and only if

$$\mathbf{0}_n \notin cl\, conv\, \{\boldsymbol{a}_t : t \in T\},\ (\mathbf{0}_n, 1)' \in O^+(cl\, G) \text{ and } \boldsymbol{c} \in int\, M.$$

(v) $int\, \Pi_i = \emptyset$ for $i = 5, 6$.

The following theorem shows that the characterization of the interior of the sets that are generated with the primal-dual partition, in the case of bounded coefficients, is like the continuous case. However, in the new case Slater condition is replaced by strong Slater condition, because in the case of bounded coefficient there are parameters that satisfy the Slater condition, but not the strong Slater condition (see Example 4).

Theorem 3.3. Let $\pi = (\boldsymbol{a}, b, \boldsymbol{c})$ a parameter with bounded coefficients. Then
(i) $\pi \in int\ \Pi_1$ if and only if π satisfies the Strong Slater condition and $\boldsymbol{c} \in int\ M$.
(ii) $\pi \in int\ \Pi_2$ if and only if there exists $\boldsymbol{y} \in \mathbb{R}^n$ such that

$$\boldsymbol{c}'\boldsymbol{y} < 0 \text{ and } \boldsymbol{a}_t'\boldsymbol{y} > 0 \text{ for all } t \in T.$$

(iii) $\pi \in int\ \Pi_3$ if and only if $(\boldsymbol{0}_n, 1)' \in int\ N$.
(iv) $int\ \Pi_i = \emptyset$ for $i = 4, 5, 6$.

The proof of the previous theorem follows from Theorems 3.1 and 3.2, and the next observation.

Observation 3.4. If $\boldsymbol{a}$ and b are bounded, then

$$O^+(cl\ G) = \{\boldsymbol{0}_{n+1}\}.$$

Note that in the case of bounded coefficients, $\pi \in int\ \Pi_3$, $M \neq \mathbb{R}^n$ which is impossible, because otherwise $(\boldsymbol{0}_n, 1)' \in O^+(cl\ G)$ (see Theorem 3.2), and in this case, $O^+(cl\ G) = \{\boldsymbol{0}_{n+1}\}$.

In the same way we get that $int\ \Pi_4 = \emptyset$.

4. First refined primal-dual partition

A refinement of the primal-dual partition follows from classifying the bounded primal and dual problems in two categories. The first one, is formed by solvable problems with bounded optimal set (S). The second one, includes unsolvable problems and those that have unbounded optimal set (N). The refinement is called refined primal-dual partition and it is shown in Table 2.

<table>
<tr><td rowspan="2">$(D)\backslash(P)$</td><td rowspan="2">IC</td><td colspan="2">B</td><td rowspan="2">UB</td></tr>
<tr><td>S</td><td>N</td></tr>
<tr><td>IC</td><td>Π_4</td><td></td><td>Π_5</td><td>Π_2</td></tr>
<tr><td>B S</td><td></td><td>Π_1^1</td><td>Π_1^3</td><td rowspan="2"></td></tr>
<tr><td>B N</td><td>Π_6</td><td>Π_1^2</td><td>Π_1^4</td></tr>
<tr><td>UB</td><td>Π_3</td><td colspan="2"></td><td></td></tr>
</table>

Table 2

In the refinement,

$$\Pi_1^1 := \Pi_S^P \cap \Pi_S^D,\ \Pi_1^2 := \Pi_S^P \cap \Pi_N^D,\ \Pi_1^3 := \Pi_N^P \cap \Pi_S^D \text{ and } \Pi_1^4 := \Pi_N^P \cap \Pi_N^D.$$

The other sets are the same as in the primal-dual partition.

According to the Duality Theorem ([3, Theorem 4.2]), in ordinary linear programming ([2] and [3]) we have $\Pi_1^i = \emptyset$ for $i = 2, 3, 4$. However, in the case of bounded coefficients, the mentioned sets are nonempty, which comes from the Theorem 3.1 in [5].

Theorem 4.1. $\Pi_1^i \neq \emptyset,\, i = 1, \ldots, 4$.

The conditions characterizing the sets generated by the refined primal-dual partition in the continuous case, are as follows:

Theorem 4.2. [5, Theorem 3.3] The following statements are true:
(i) $\pi \in \Pi_1^1$ if and only if $\boldsymbol{c} \in int\ M$ and π satisfies the Slater condition;
(ii) $\pi \in \Pi_1^2$ if and only if $(\boldsymbol{0}_n, 1)' \notin cl\ K$, $\boldsymbol{c} \in int\ M$ and π does not satisfy the Salter condition;
(iii) $\pi \in \Pi_1^3$ if and only if $\boldsymbol{c} \in M \setminus int\ M$ and π satisfies the Salter condition;
(iv) $\pi \in \Pi_1^4$ if and only if $(\boldsymbol{0}_n, 1)' \notin cl\ K$, $\boldsymbol{c} \in M \setminus int\ M$ and π does not satisfy the Slater condition.

The condition that characterizes the set Π_S^P and which is presented in Lemma 2.1 is true in the case of bounded coefficients. In the following example, $\boldsymbol{c}^1 \in M_1$ and π^1 satisfies the Slater condition. However, the dual problem associated with the parameter π^1 is unsolvable. Hence, we show that the condition that characterizes the set Π_S^D fails in the case of bounded coefficients.

Example 1. Let $T = [0, 1]$ and $n = 2$. We define $\pi^1 := (\boldsymbol{a}^1, b^1, \boldsymbol{c}^1)$ such that, $\boldsymbol{a}_t^1 := (t, 1)'$ for all $t \in T$,

$$b_t^1 := \begin{cases} 1, & if\ t = 0 \\ 0, & if\ 0 < t < 1 \\ -1, & if\ t = 1, \end{cases}$$

and $\boldsymbol{a}_t^1 := \left(\frac{1}{3}, 1\right)'$. In [7] it is shown that $\boldsymbol{c}^1 \in M_1$ and π^1 satisfies the Salter condition, but the dual problem is not solvable. In the above example, the parameter has unsolvable dual problem, which means that the optimal set of the dual problem is empty, and in particular it is bounded. In addition, the parameter satisfies the strong Slater condition, in fact, for $\varepsilon = \frac{1}{2}$ the point $(0, 2)'$ is a strong Slater point. This leads to the following conjecture.

Conjecture 4.3. If $\boldsymbol{c} \in M$ and π satisfies the strong Salter condition, then Λ^* is bounded.

Resolving the conjecture above requires a result that is analogous to Carathéodory's Theorem for positive linear combinations [9, Corollary 17.1.2].

Lemma 4.4. If $\sum_{t\in T} \lambda_t \boldsymbol{a}_t = \boldsymbol{c}$ with $\lambda_t \geq 0$ for all $t \in T$, then $\sum_{i=1}^{n+1} \gamma_i \boldsymbol{a}_{t_i} = \boldsymbol{c}$ and $\sum_{t\in T} \lambda_t = \sum_{i=1}^{n+1} \gamma_i$.

Proof. Let $\sum_{t\in T} \lambda_t \boldsymbol{a}_t = \boldsymbol{c}$. The case $\sum_{t\in T} \lambda_t = 0$ is obvious.

Now, if $\sum_{t\in T} \lambda_t = \lambda > 0$, then $\sum_{t\in T} \frac{\lambda_t}{\lambda} \boldsymbol{a}_t = \frac{\boldsymbol{c}}{\lambda}$ and $\sum_{t\in T} \frac{\lambda_t}{\lambda} = 1$. From the Carathéodory Theorem for convex combinations ([9, Theorem 17.1]), $\sum_{i=1}^{n+1} \beta_i \boldsymbol{a}_{t_i} = \frac{\boldsymbol{c}}{\lambda}$, where $\sum_{i=1}^{n+1} \beta_i = 1$. Multiplying by λ the above equalities yields $\sum_{i=1}^{n+1} \lambda\beta_i \boldsymbol{a}_{t_i} = \boldsymbol{c}$ and $\sum_{i=1}^{n+1} \lambda\beta_i = \lambda$. If we make $\gamma_i = \lambda\beta_i$, we conclude that $\sum_{i=1}^{n+1} \gamma_i \boldsymbol{a}_{t_i} = \boldsymbol{c}$ and $\sum_{t\in T} \lambda_t = \sum_{i=1}^{n+1} \gamma_i$.

The last lemma shows that all linear positive combination can be represented by $n + 1$ elements and that the sums of the coefficients are equal. The next theorem proves the conjecture.

Theorem 4.5. Let π a parameter, with $|T| \geq n + 2$. If $\boldsymbol{c} \in M$ and π satisfies the strong Slater condition, then Λ^* is bounded with respect to the norm l_1.

Proof. As $\boldsymbol{c} \in M$, then $\pi \in \Pi_C^D$. In addition, as π satisfies the strong Slater condition, $\pi \in \Pi_C^P$. This means that, $\pi \in \Pi_1$, in particular, $\pi \in \Pi_A^D$.

Now, if $\Lambda^* = \emptyset$, the result is obvious. Consider that $\Lambda^* \neq \emptyset$ and Λ^* is not bounded. Let $\{\lambda^m\}$ in Λ^* such that, $\sum_{t\in T} \lambda_t^m \to \infty$. Since $\Lambda^* \subseteq \Lambda$ and, for all m, λ^m is in Λ^*, we have

$$\sum_{t\in T} \lambda_t^m \begin{pmatrix} \boldsymbol{a}_t \\ b_t \end{pmatrix} = \begin{pmatrix} \boldsymbol{c} \\ v^D(\pi) \end{pmatrix}.$$

From the equality above we have that for each m, $(\boldsymbol{c}, v^D(\pi))'$ is a positive linear combination of $\{(\boldsymbol{a}_t, b_t)',\ t \in T\}$. From the previous Lemma we have, for all m,

$$\sum_{i=1}^{n+2} \beta_i^m \begin{pmatrix} \boldsymbol{a}_{t_i}^m \\ b_{t_i}^m \end{pmatrix} = \begin{pmatrix} \boldsymbol{c} \\ v^D(\pi) \end{pmatrix}$$

with

$$\sum_{i=1}^{n+2} \beta_i^m = \sum_{t\in T} \lambda_t^m.$$

Then

$$(1) \qquad \sum_{i=1}^{n+2} \frac{\beta_i^m}{\sum_{i=1}^{n+2} \beta_i^m} \begin{pmatrix} \boldsymbol{a}_{t_i}^m \\ b_{t_i}^m \end{pmatrix} = \frac{1}{\sum_{i=1}^{n+2} \beta_i^m} \begin{pmatrix} \boldsymbol{c} \\ v^D(\pi) \end{pmatrix}.$$

Since $\left\{\frac{\beta_i^m}{\sum_{i=1}^{n+2} \beta_i^m}\right\}$ and $\left\{\left(\boldsymbol{a}_{t_i}^m, b_{t_i}^m\right)'\right\}$ are bounded, if $m \to \infty$ in (1), it follows that

$$\sum_{i=1}^{n+2} \beta_i \begin{pmatrix} \boldsymbol{a}_i \\ b_i \end{pmatrix} = \begin{pmatrix} \boldsymbol{0}_n \\ 0 \end{pmatrix}$$

with

$$\sum_{i=1}^{n+2} \beta_i = 1$$

Then

$$\begin{pmatrix} \boldsymbol{0}_n \\ 0 \end{pmatrix} \in cl\ conv \left\{ \begin{pmatrix} \boldsymbol{a}_t \\ b_t \end{pmatrix}, t \in T \right\},$$

this means that the strong Slater condition fails in this case, which contradicts the hypothesis.

Corollary 4.6. If Λ^* is not bounded with the norm l_1 and $|T| \geq n + 2$, then π does not satisfy the strong Slater condition.

Observation 4.7. If Λ^* is bounded with the norm l_1, then it is bounded with the norm l_∞ too. In fact, if there is $M \in \mathbb{R}_{++}$ such that, $\|\lambda\|_1 \leq M$ for all $\lambda \in \Lambda^*$, then

$$M \geq \|\lambda\|_1 = \sum_{t \in T} \lambda_t \geq \max_{t \in T} \lambda_t = \|\lambda\|_\infty.$$

In this way, we have that Λ^* is bounded with the norm l_∞.

Observation 4.8. If $|T| < n + 2$, we have a parameter in finite case.

The above theorem implies, that under certain conditions

$$sup\, \{\|\lambda\|_1, \lambda \in \Lambda^*\} < \infty.$$

In the next theorem we show that if $\boldsymbol{c} = \boldsymbol{0}_n$, then

$$sup\, \{\|\lambda\|_1, \lambda \in \Lambda^*\} = 0.$$

Theorem 4.9. Let $\pi = (\boldsymbol{a}, b, \boldsymbol{c})$ a parameter with $\boldsymbol{c} = \boldsymbol{0}_n$ and $|T| \geq n + 2$. If $\boldsymbol{c} \in int\ M$ and π satisfies the strong Slater condition, then $\Lambda^* = \{\lambda \equiv 0\}$.

Proof. First, since $\boldsymbol{c} \in M$ and π satisfies the strong Slater condition, Theorem 4.5 implies that Λ^* is bounded. On the other hand, since the primal problem is consistent and $\boldsymbol{c} \in int\ M$, we have that $v^P(\pi) = v^D(\pi)$ [4, Theorem 8.1]. Moreover, since $\boldsymbol{c} = \boldsymbol{0}_n$, $v^D(\pi) = 0$.

Now, suppose that $\Lambda^* = \emptyset$. In this case, the function $\lambda \equiv 0$ is an optimal solution, which contradicts the assumption.

Since $\Lambda^* \neq \emptyset$, let $\lambda \in \Lambda^*$ and suppose that $\sum_{t \in T} \lambda_t > 0$. Because $\boldsymbol{c} = \boldsymbol{0}_n$, we have

$$\sum_{t \in T} \lambda_t \begin{pmatrix} \boldsymbol{a}_t \\ b_t \end{pmatrix} = \begin{pmatrix} \boldsymbol{0}_n \\ 0 \end{pmatrix}.$$

Then

$$\sum_{t \in T} \frac{\lambda_t}{\sum_{t \in T} \lambda_t} \begin{pmatrix} \boldsymbol{a}_t \\ b_t \end{pmatrix} = \begin{pmatrix} \boldsymbol{0}_n \\ 0 \end{pmatrix}$$

and

$$\sum_{t \in T} \frac{\lambda_t}{\sum_{t \in T} \lambda_t} = 1,$$

therefore

$$\begin{pmatrix} \boldsymbol{0}_n \\ 0 \end{pmatrix} \in conv \left\{ \begin{pmatrix} \boldsymbol{a}_t \\ b_t \end{pmatrix}, t \in T \right\} \subset cl\ conv \left\{ \begin{pmatrix} \boldsymbol{a}_t \\ b_t \end{pmatrix}, t \in T \right\},$$

this means that π does not satisfy the strong Slater condition, which is a contradiction. Thus

$$\sum_{t \in T} \lambda_t = 0$$

Since λ was arbitrary, we conclude that $\Lambda^* = \{\lambda \equiv 0\}$.

Now, we present another proof of the above theorem.

Proof. The first part of the last proof shows that Λ^* is bounded, $\Lambda^* \neq \emptyset$ and $v^D(\pi) = 0$. Now, suppose there is $\lambda \in \Lambda^*$ such that, $\sum_{t \in T} \lambda_t > 0$. If $\delta > 0$, we have $\delta\lambda \in \Lambda^*$. In fact, since $\lambda \in \Lambda^*$ and $\boldsymbol{c} = \boldsymbol{0}_n$ we have

$$\sum_{t \in T} \lambda_t \binom{\boldsymbol{a}_t}{b_t} = \binom{\boldsymbol{0}_n}{0},$$

which implies that

$$\sum_{t \in T} \delta\lambda_t \binom{\boldsymbol{a}_t}{b_t} = \delta \sum_{t \in T} \lambda_t \binom{\boldsymbol{a}_t}{b_t} = \delta \binom{\boldsymbol{0}_n}{0} = \binom{\boldsymbol{0}_n}{0}.$$

On the other hand, $\|\delta\lambda\|_1 = \delta\|\lambda\|_1 > 0$. If $\delta \to \infty$, also $\|\delta\lambda\|_1 \to \infty$. This means that Λ^* is not bounded, which contradicts Theorem 4.5.

In the case when $\boldsymbol{c} \neq \boldsymbol{0}_n$ we shall only demonstrate that the infimum is strictly positive.

Theorem 4.10. Let $\pi = (\boldsymbol{a}, b, \boldsymbol{c})$ a parameter with $\boldsymbol{c} \neq \boldsymbol{0}_n$ such that, $\boldsymbol{c} \in M$ and π satisfies the strong Slater condition, then

$$inf\ \{\|\lambda\|_1 : \lambda \in \Lambda^*\} > 0.$$

Proof. Suppose that

$$inf\ \{\|\lambda\|_1 : \lambda \in \Lambda^*\} = 0.$$

Then there is $\{\lambda^m\}$ in Λ^* such that,

$$\{\lambda^m\} \to 0,$$

i.e.,

$$\sum_{t \in T} \lambda_t^m \to 0.$$

Since $\lambda^m \in \Lambda^*$, for each m, we have

$$\sum_{t \in T} \lambda_t^m \binom{\boldsymbol{a}_t}{b_t} = \binom{\boldsymbol{c}}{v^D(\pi)}.$$

Lemma 4.4 implies

$$\text{(2)} \qquad \sum_{i=1}^{n+2} \gamma_{t_i}^m \binom{\boldsymbol{a}_{t_i}}{b_{t_i}} = \binom{\boldsymbol{c}}{v^D(\pi)} \ and \ \sum_{i=1}^{n+2} \gamma_{t_i}^m = \sum_{t \in T} \lambda_t^m.$$

Now, because $\sum_{t \in T} \lambda_t^m \to 0$ and also $\left\{\left(\boldsymbol{a}_{t_i}, b_{t_i}\right)'\right\}$ and $\{\gamma_{t_i}^m\}$ are bounded sequences, making $m \to \infty$ in (2) it yields

$$\sum_{i=1}^{n+2} 0 \binom{\boldsymbol{a}_i}{b_i} = \binom{\boldsymbol{c}}{v^D(\pi)}.$$

The above equality implies $\boldsymbol{c} = \boldsymbol{0}_n$, which is a contradiction.

With the examples below we will show that the conditions that characterize the sets generated by the refined primal-dual partition, in the continuous case, do not hold in the case of bounded coefficients.

We consider again the parameter of Example 1, and we show that $\boldsymbol{c} \in int\ M$ and that π satisfying the strong Slater condition are not sufficient conditions for $\pi \in \Pi_1^1$.

Example 2. The primal problem is:

$$
\begin{aligned}
P_2: \ & \inf \frac{1}{3}x_1 + x_2 \\
s.t. \ & \quad x_2 \geq 1 \\
& tx_1 + x_2 \geq 0 \quad t \in (0,1) \\
& x_1 + x_2 \geq -1.
\end{aligned}
$$

In [8] it is shown that $\boldsymbol{c}_2 \in int\ M_2$ and that π_2 satisfies the Slater condition (in particular, $(0,2)'$ is a Slater point). Now, for $\varepsilon = \frac{1}{2}$, $(0,2)'$ is a strong Slater point. The latter would imply, in the continuous case, that $\pi^2 \in \Pi_1^1$. However, in [7] it is shown that $v^P(\pi^2) = \frac{2}{3}$ and $F_2^* = \{(-1,1)'\}$, but the dual problem

$$
\begin{aligned}
D_2: \ & \sup \lambda_0 - \lambda_1 \\
s.t.\ & \lambda_0 \binom{0}{1} + \sum_{t \in (0,1)} \lambda_t \binom{t}{1} + \lambda_1 \binom{1}{1} = \binom{1/3}{1} \\
& \lambda \in \mathbb{R}_+^{(T)}
\end{aligned}
$$

is not solvable. Therefore, $\pi^2 \notin \Pi_1^1$ in the case of bounded coefficients.

The following example shows that if $\pi \in \Pi_1^1$, it is not necessary condition that $\boldsymbol{c} \in int\ M$ and that π satisfy the strong Slater condition.

Example 3. Let $\alpha > 0$ and consider the following problem in $\mathbb{R}$:

$$
\begin{aligned}
P_3: \ & \inf \alpha x_1 \\
s.t.\ & tx_1 \geq t^2 \quad t \in (0,1].
\end{aligned}
$$

The problem is solvable, with $F_3 = \{x_1 : x_1 \geq 1\}$, $v^P(\pi^3) = \alpha$ y$F^* = \{x_1 = 1\}$. Also,

$$\boldsymbol{c}^3 = \alpha \in int\ M_3 = int\ (cone\ \{t : t \in (0,1]\}) = \mathbb{R}_{++}.$$

On the other hand, $(0,0)' \in cl\ conv\ \{(t,t^2)' : t \in (0,1]\}$, which implies that π^3 does not satisfy the strong Slater condition.

The dual problem is:

$$
\begin{aligned}
D_3: \ & \sup \sum_{t \in (0,1]} \lambda_t t^2 \\
s.t.\ & \sum_{t \in (0,1]} \lambda_t t = \alpha \\
& \lambda \in \mathbb{R}_+^{(T)}.
\end{aligned}
$$

The function

$$\lambda_t = \begin{cases} \alpha, & if\ t = 1 \\ 0, & if\ 0 < t < 1, \end{cases}$$

is the only feasible solution and it is optimal. We conclude that D_3 is solvable with bounded optimal set. This way, we have a parameter $\pi^3 \in \Pi_1^1$, where $\boldsymbol{c}^3 \in int\ M_3$, but π^3 does not satisfy the strong Slater condition.

In the particular case $\boldsymbol{c} = \boldsymbol{0}_n$ we present a sufficient condition which implies that a given parameter π belongs to the set Π_1^1. The proof follows from Lemma 2.1 and Theorem 4.9.

Corollary 4.11. Let $\pi = (\boldsymbol{a}, b, \boldsymbol{c})$ be a parameter with $\boldsymbol{c} = 0_n$ and $|T| \geq n + 2$. If $\boldsymbol{c} \in int\ M$ and π satisfies the strong Slater condition, then $\pi \in \Pi_1^1$.

As a consequence of the above corollary, we have that the feasible set of the system $\{a_t x \geq b_t : t \in T\}$ (when $|T| \geq n + 2$) is nonempty and bounded, if $0_n \in int\ cone\ \{a_t : t \in T\}$ and there are $\varepsilon > 0$ and

$\bar{\boldsymbol{x}} \in \mathbb{R}^n$ such that, $a_t\bar{\boldsymbol{x}} \geq b_t + \varepsilon$ for all $t \in T$. In fact, if the conditions are true and we consider the parameter $\pi = (\boldsymbol{a}, b, \boldsymbol{0}_n)$, then $\pi \in \Pi_1^1$. In particular, F^* is nonempty and bounded. Since in this case $F = F^*$ the result is immediate.

Corollary 4.12. Let $\pi = (\boldsymbol{a}, b, \boldsymbol{c})$ be a parameter with $b \equiv 0$ and $|T| \geq n + 2$. If $\boldsymbol{c} \in int\ M$ and π satisfies the strong Slater condition, then $\pi \in \Pi_1^1$.

Proof. If $\boldsymbol{c} \in M$ and π satisfies the strong Slater condition, then, by Theorem 4.5, Λ^* is bounded. Now, since $\boldsymbol{c} \in M$ we have that $\Lambda \neq \emptyset$. Furthermore, every feasible solution of the dual problem is optimal because $b \equiv 0$. In this way, we have $\Lambda^* \neq \emptyset$. This implies that $\pi \in \Pi_S^D$. It only remains to prove that $\pi \in \Pi_S^P$, but this one is equivalent to $\boldsymbol{c} \in int\ M$, if the parameter π has consistent primal problem ([4, Corollary 9.3.1]). As $\boldsymbol{c} \in int\ M$, we will only show that π has consistent primal problem. But it is consistent because, by hypothesis, π satisfies the strong Slater condition.

The parameter π^3 presented in Example 3 shows us that the condition presented in Theorem 4.2(ii), for the set Π_1^2, fails in the case of bounded coefficient. In particular, we show that $(0,1)' \notin cl\ N$, $\boldsymbol{c} \in int\ M$ and π without the strong Slater condition are not sufficient conditions for π to belong to Π_1^2. In fact, $(0,1)' \notin cl\ N_3$ because the primal problem is consistent, also $\boldsymbol{c}^3 \in int\ M_3$ and π^3 does not satisfy the strong Salter condition, but $\pi^3 \notin \Pi_1^2$.

The parameter π^2 of Example 2, also shows that $(0,1)' \notin cl\ N$, $\boldsymbol{c} \in int\ M$ and π without the strong Slater condition are not necessary conditions for the belonging of π to Π_1^2. In fact, $\pi^2 \in \Pi_1^2$, $(0,1)' \notin cl\ N_2$ and $\boldsymbol{c}^2 \in int\ M_2$, but π^2 satisfies the Slater condition.

With the next example we demonstrate that the condition presented in Theorem 4.2 (iii) is not valid in the case of bounded coefficients. In particular, it shows that $\boldsymbol{c} \in M \setminus int\ M$ and π with the strong Slater condition are not necessary conditions for $\pi \in \Pi_1^3$.

Example 4. Consider the problem in $\mathbb{R}^2$ defined by:

$$\begin{aligned} P_4:\ & \inf x_1 + x_2 \\ & s.t.\ \ t^2x_1 + tx_2 \geq t,\ \ t \in (0,1]. \end{aligned}$$

The feasible set is shown in Figure 1.

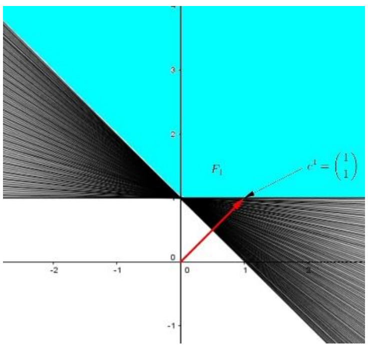

Figure 1. Feasible set of P_4.

From the above figure, we have,

$$v^P(\pi^4) = 1$$

and

$$F_4^* = \{(x_1, x_2)' : x_2 = 1 - x_1, x_1 \leq 0\}.$$

The problem is solvable with unbounded optimal set. On the other hand, it is true that

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in cl\ conv \left\{ \begin{pmatrix} t^2 \\ t \\ t \end{pmatrix} : t \in (0, 1] \right\},$$

which implies that π^4 does not satisfy the strong Slater condition. The figure 2 shows us that

$$\boldsymbol{c}^4 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in M_4 \backslash int\ M_4.$$

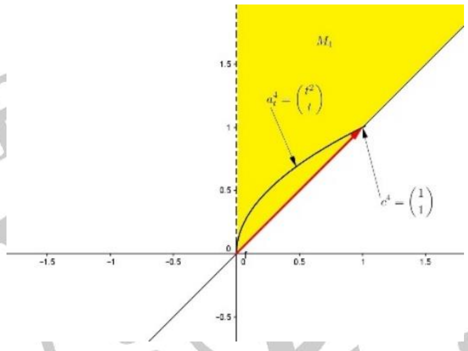

Figure 2. Cone M_4 of P_4

The dual problem is:

$$\begin{aligned} D_4: \quad & \sup \sum_{t \in (0,1]} \lambda_t t \\ s.t. \quad & \sum_{t \in (0,1]} \lambda_t \begin{pmatrix} t^2 \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ & \lambda \in \mathbb{R}_+^{(T)}. \end{aligned}$$

The function

$$\lambda = \begin{cases} 1, & si\ t = 1 \\ 0, & si\ 0 < t < 1, \end{cases}$$

is the only feasible solution and it is also optimal. It implies that D_4 is solvable and it has bounded optimal set. This way, $\pi^4 \in \Pi_1^3$ and although $\boldsymbol{c}^4 \in M_4 \backslash int\ M_4$. we have that π^4 does not satisfy the strong Slater condition.

With the following example we show that the conditions $\boldsymbol{c} \in M \backslash int\ M$ and π satisfying the strong Slater condition are not sufficient for $\pi \in \Pi_1^3$.

Example 5. Consider the following problem in $\mathbb{R}^2$:

$$\begin{aligned} P_5: \quad & \inf x_2 \\ s.t. \quad & x_2 \geq t, \quad t \in [0,1) \\ & x_1 \geq 0. \end{aligned}$$

The feasible set is shown in Figure 3. We have that

$$v^P(\pi^5) = 1$$

and

$$F_5^* = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_2 = 1, x_1 \geq 0 \right\}.$$

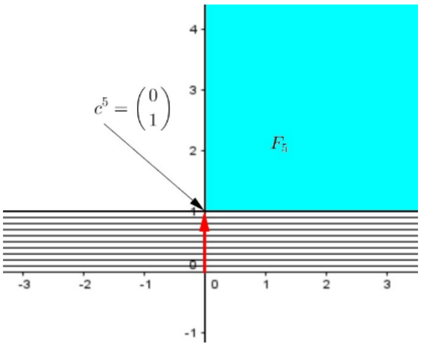

Figure 3. Feasible set of P_5

The problem is solvable with unbounded optimal set. On the other hand, π^5 satisfies the strong Slater, in fact, let $\varepsilon = 1$ and consider $\bar{x} = (3, 3)'$. Now, Figure 4 it shows us that $\boldsymbol{c}^5 = (0, 1)' \in M_5 \backslash int\ M_5$.

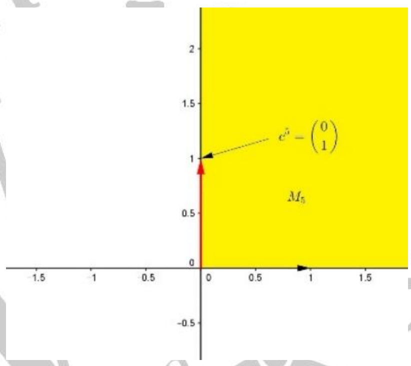

Figure 4. Cone M_5 of P_5

The dual problem is:

$$D_5: \quad \sup \sum_{t \in [0,1)} \lambda_t t$$

$$s.t. \quad \sum_{t \in [0,1)} \lambda_t \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\lambda \in \mathbb{R}_+^{(T)}.$$

From the constraints set, it follows $\sum_{t \in [0,1)} \lambda_t = 1$ and $\lambda_1 = 0$. Let us consider the sequence of feasible points $\bar{\theta}^m = (\lambda^m; \lambda_1)$, where $\lambda_1 = 0$ and λ^m is defined by

$$\lambda_t^m = \begin{cases} 1, & if\ t = 1 - \dfrac{1}{m} \\ 0, & if\ t \in [0,1) \backslash \dfrac{1}{m}. \end{cases}$$

It follows that

$$\sum_{t \in [0,1)} \lambda_t^m t = 1 - \frac{1}{m}$$

approaches to 1 when m tends to infinity. This implies that, $v^D(\pi^5) = 1$, but $\Lambda^* = \emptyset$, this is, the dual problem is not solvable. So, it has a parameter π^5 in which $\boldsymbol{c}^5 \in M_5 \backslash int\ M_5$ and π^5 satisfies the strong Slater condition, but for $\pi^5 \notin \Pi_1^3$.

In the previous example $\pi^5 \in \Pi_1^4$, $(\boldsymbol{0}_n, 1)' \notin cl\ N_5$ and $\boldsymbol{c}^5 \in M_5 \backslash int\ M_5$, but π^5 does not satisfy the strong Slater condition. This tells us that the condition presented in Theorem 4.2(iv), for the set Π_1^4 does not remain valid in the case of bounded coefficients. In particular, with the example we show that $(\mathbf{0}_n, 1)' \notin cl\, N$, $c \in M\backslash int\ M$ and that π does not satisfy the strong Slater condition are not necessary conditions for $\pi \in \Pi_1^4$.

In Example 4, $(\mathbf{0}_n, 1)' \notin cl\, N_4$, $\boldsymbol{c}^4 \in M_4\backslash int\ M_4$ and π^4 does not satisfy the strong Slater condition, but $\pi^4 \notin \Pi_1^4$. Thus, we show that $(\mathbf{0}_n, 1)' \notin cl\, N$, $\boldsymbol{c} \in M\backslash int\ M$ and π without the strong Slater condition are not sufficient for π to belong to Π_1^4.

Following similar arguments to those in [5] we have that $\Pi_5^1 = \emptyset$ holds in the case of bounded coefficients. We have thus that $\Pi_5 = \Pi_N^P \cap \Pi_{IC}^D$.

With the following example we see that $\Pi_6^1 = \Pi_{IC}^P \cap \Pi_S^D \neq \emptyset$ in the case of bounded coefficients, on the contrary of what happens in the continuous case.

Example 6. Consider the problem in $\mathbb{R}^2$ given by:

$$P_6: \quad \inf 0$$
$$s.t. \;\; tx_1 + tx_2 \geq 1, \quad t \in (0,1].$$

The problem is inconsistent, because

$$\begin{pmatrix}0\\0\\1\end{pmatrix} = cl\ cone\left\{\begin{pmatrix}t\\t\\1\end{pmatrix}, t \in (0,1]\right\}.$$

The dual problem is:

$$D_6: \quad sum \sum_{t\,\in\,(0,1]} \lambda_t$$
$$s.t. \quad \sum_{t\,\in\,(0,1]} \lambda_t \begin{pmatrix}t\\t\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$
$$\lambda \in \mathbb{R}_+^{(T)}.$$

From the system of restrictions, we have

$$\sum_{t\,\in\,(0,1]} \lambda_t t = 0.$$

Since $t \in (0,1]$, the above equality is only possible if $\sum_{t\,\in\,(0,1]} \lambda_t = 0$. It follows that $v^D(\pi^6) = 0$ and $\Lambda^* = \{\lambda \equiv \bar{0}\}$. Therefore, D_6 is bounded and solvable with bounded optimal set.

The set Π_6^2 is also nonempty. This could be seen if we look at the continuous case.

This section ends with the presentation of several necessary conditions.

The following two result are obtained from [4, Corollary 9.3.1] and Corollary 4.6.

- If $\pi \in \Pi_1^2$ and Λ^* is unbounded, then $\boldsymbol{c} \in int\ M$ and π does not satisfy the strong Slater condition.
- If $\pi \in \Pi_1^4$ and Λ^* is unbounded, then $\boldsymbol{c} \in M\backslash int\ M$ and π do not satisfy the strong Slater condition.

The next result is obtained from [7].

- If $\pi \in \Pi_1^2$ and $\Lambda^* = \emptyset$, then there exist $\{\pi^r\}$ in Π such that, $\pi^r \to \pi$, $\boldsymbol{c}^r \in int\ M_r$ and π^r satisfies the strong Slater condition.

Conclusions

We conclude mentioning that we have obtained a sufficient condition for the boundedness of the optimal set, which might however be empty, of the dual problem. Conditions that guarantee the solvability of the dual problem turns out to be complicated task even in the continuous case. This could be a challenge problem for a future work.

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