Primal-dual and general primal-dual partitions in linear semi-infinite programming with bounded coefficients

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Contenido

Preliminaries

The primal problem is defined as:

$$P: \quad \begin{array}{ll} \text{ínf} & \boldsymbol{c}'\boldsymbol{x} \\ s.t. & \boldsymbol{a}_t'\boldsymbol{x} \geq b_t, \ t \in T \end{array}$$

where $\boldsymbol{c} \in \mathbb{R}^n$, T is a nonempty fixed index set, $\boldsymbol{a} : T \to \mathbb{R}^n$, whit $\boldsymbol{a}(t) := (a_1(t), ..., a_n(t)) := \boldsymbol{a}_t'$ and $b : T \to \mathbb{R}$, with $b(t) := b_t$. Furthermore, $\boldsymbol{a}$ and b are bounded.

First and second moment cones

$$M := cone\,\{\boldsymbol{a}_t,\ t \in T\}, \qquad N := cone\left\{\begin{pmatrix}\boldsymbol{a}_t\\ b_t\end{pmatrix},\ t \in T\right\}.$$

Characteristic cone

$$K := cone\left\{\begin{pmatrix}\boldsymbol{a}_t\\ b_t\end{pmatrix},\ t \in T; \begin{pmatrix}\boldsymbol{0}_n\\ -1\end{pmatrix}\right\}.$$

The dual problem of P is:

$$\begin{aligned} D: \quad \sup \; & \sum_{t\in T} \lambda_t b_t \\ s.t. \quad & \sum_{t\in T} \lambda_t \boldsymbol{a}_t = \boldsymbol{c} \\ & \lambda \in \mathbb{R}_+^{(T)}. \end{aligned}$$

The pair of problems

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$$s.t. \; \boldsymbol{a}_t'\boldsymbol{x} \geq b_t, \; t \in T \qquad\qquad s.t. \sum_{t \in T} \lambda_t \boldsymbol{a}_t = \boldsymbol{c}$$

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$$\lambda \in \mathbf{R}_+^{(T)}$$

is called primal-dual pair.

This pair is represented by the triplet $\pi := (\boldsymbol{a}, \; b, \; \boldsymbol{c})$.

Definition

Let $\pi = (\boldsymbol{a}, b, \boldsymbol{c}) \in \Pi$. π satisfies the **Slater condition** if there exist $\overline{\boldsymbol{x}} \in \mathbb{R}^n$, such that,

$$\boldsymbol{a}_t^{\prime} \overline{\boldsymbol{x}} > b_t \text{ for all } t \in T.$$

Definition

Let $\pi = (\boldsymbol{a}, b, \boldsymbol{c}) \in \Pi$. π satisfies the **strong Slater condition** if there are $\epsilon > 0$ and $\overline{\boldsymbol{x}} \in \mathbb{R}^n$, such that,

$$\boldsymbol{a}_t^{\prime} \overline{\boldsymbol{x}} \geq b_t + \epsilon \text{ for all } t \in T.$$

The parameters space is:

$$\Pi \; := \; \mathbb{B}(T)^n \times \mathbb{B}(T) \times \mathbb{R}^n.$$

equipped with the pseudometrics $d :\to [0, +\infty)$, defined by

$$d(\pi^1, \pi^2) := \text{máx}\left\{ \|\boldsymbol{c}^1 - \boldsymbol{c}^2\|, \sup_{t \in T} \left\| \begin{pmatrix} \boldsymbol{a}_t^1 \\ b_t^1 \end{pmatrix} - \begin{pmatrix} \boldsymbol{a}_t^2 \\ b_t^2 \end{pmatrix} \right\| \right\}.$$

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- $\Pi_{UB}^{P} := \{\pi \in \Pi \mid F \neq \varnothing \text{ and } v^{P}(\pi) = -\infty\}$

- $\Pi_{IC}^{D} := \{\pi \in \Pi \mid \Lambda = \varnothing\}$
- $\Pi_{B}^{D} := \{\pi \in \Pi \mid \Lambda \neq \varnothing \text{ and } v^{D}(\pi) < \infty\}$
- $\Pi_{UB}^{D} := \{\pi \in \Pi \mid \Lambda \neq \varnothing \text{ and } v^{D}(\pi) = \infty\}$

Primal-dual partition

$(D) \setminus (P)$	IC	B	UB
IC	Π_4	Π_5	Π_2
B	Π_6	Π_1	
UB	Π_3		

Example, $\Pi_1 := \Pi_B^P \cap \Pi_{UB}^D$.

Theorem

(i) $\pi \in \Pi_1$ *if and only if* $(\mathbf{0}_n, 1)' \notin \mathit{cl}\ N$ *and* $\mathbf{c} \in M$.

(ii) $\pi \in \Pi_2$ *if and only if* $(\mathbf{0}_n, 1)' \notin \mathit{cl}\ N$ *and* $(\{\mathbf{c}\} \times \mathbb{R}) \cap \mathit{cl}\ N = \varnothing$.

(iii) $\pi \in \Pi_3$ *if and only if* $\{\mathbf{c}\} \times \mathbb{R} \subseteq K$.

(iv) $\pi \in \Pi_4$ *if and only if* $(\mathbf{0}_n, 1)' \in \mathit{cl}\ N$ *and* $\mathbf{c} \notin M$.

(v) $\pi \in \Pi_5$ *if and only if* $\mathbf{c} \notin M$, $(\mathbf{0}_n, 1)' \notin \mathit{cl}\ N$ *and* $(\{\mathbf{c}\} \times \mathbb{R}) \cap \mathit{cl}\ N \neq \varnothing$.

(vi) $\pi \in \Pi_6$ *if and only if* $(\mathbf{0}_n, 1)' \in \mathit{cl}\ N$, $\mathbf{c} \in M$ *and* $\{\mathbf{c}\} \times \mathbb{R} \nsubseteq K$.

Stability of the primal-dual partition

$(D) \setminus (P)$	IC	B	UB
IC	Π_4	Π_5	Π_2
B	Π_6	Π_1	
UB	Π_3		

Theorem

Let $\pi = (\boldsymbol{a}, b, \boldsymbol{c})$ a parameter with bounded coefficients. The following statements are true:

(i) $\pi \in$ *int* Π_1 *if and only if* π *satisfies the strong Slater condition and* $\boldsymbol{c} \in$ *int* M*;*

(ii) $\pi \in$ *int* Π_2 *if and only if there exist* $\boldsymbol{y} \in \mathbb{R}^n$ *y* $\delta > 0$ *such that,*

$$\boldsymbol{c}^{'} \boldsymbol{y} < 0 \quad \textit{and} \quad \boldsymbol{a}_t^{'} \boldsymbol{y} \geq \delta \textit{ for all } t \in T;$$

(iii) $\pi \in$ *int* Π_3 *if and only if* $(\boldsymbol{0}_n, 1)^{'} \in$ *int* N*;*

(iv) *int* $\Pi_i = \varnothing$ *for* $i = 4, 5, 6$.

Primer refinement

$(D) \setminus (P)$	IC	B	UB
IC	Π_4	Π_5	Π_2
B	Π_6	Π_1	
UB	Π_3		

- $\Pi_S^P := \{\pi \in \Pi_A^P \mid F^* \neq \varnothing \text{ and } F^* \text{ is bounded}\}$

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First primal-dual refined partition

$D \setminus P$	IC	B: S	B: N	UB
IC	Π_4		Π_5	Π_2
B: S		Π_1^1	Π_1^3	
B: N	Π_6	Π_1^2	Π_1^4	
UB	Π_3			

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(i) $\pi \in \Pi_1^1$ *if and only if* $\boldsymbol{c} \in int\ M$ *and* π *satisfies the Slater condition;*

(ii) $\pi \in \Pi_1^2$ *if and only if* $\binom{0_n}{1} \notin cl\ K$, $\boldsymbol{c} \in int\ M$ *and* π *not satisfy the Slater condition;*

(iii) $\pi \in \Pi_1^3$ *if and only if* $\boldsymbol{c} \in M \setminus int\ M$ *and* π *satisfies the Slater condition;*

(iv) $\pi \in \Pi_1^4$ *if and only if* $\binom{0_n}{1} \notin cl\ K$, $\boldsymbol{c} \in M \setminus int\ M$ *and* π *not satisfy the Slater condition.*

Goberna M.A. and Todorov M.I.

$\pi \in \Pi_S^P$ if and only if $\binom{\mathbf{0}_n}{1} \notin cl\ K$ and $\boldsymbol{c} \in int\ M$.

$\pi \in \Pi_S^D$ if and only if $\boldsymbol{c} \in M$ and π satisfies the Slater condition.

$\boldsymbol{c} \in M$ and π satisfies the Slater condition, but $\pi \notin \Pi_S^D$.

Example 1.

$$P_1: \qquad \inf \ \tfrac{1}{3}x_1 + x_2$$

$$s.t. \quad \begin{array}{l} x_2 \geq 1 \\ tx_1 + x_2 \geq 0, \quad t \in (0,1), \\ x_1 + x_2 \geq -1 \end{array}$$

$(\frac{1}{3}, 1)' \in int\ M_1 \subset M_1$ and π^1 satisfies the Slater condition ($(0,2)$ is a Slater point). Furthermore, $v^p(\pi^1) = \frac{2}{3}$ and $F_1^* = \{(-1,1)\}$.

The dual problem

$$\begin{array}{rc} D_1: & \sup \lambda_0 - \lambda_1 \\ s.t. & \lambda_0 \binom{0}{1} + \sum_{t \in (0,1)} \lambda_t \binom{t}{1} + \lambda_1 \binom{1}{1} = \binom{\frac{1}{3}}{1} \\ & \lambda \in \mathbb{R}_+^{(T)}. \end{array}$$

is not solvable.

π^1 satisfies the strong Slater condition, in fact, for $\epsilon = \frac{1}{2}$, a strong Slater point is $\binom{0}{2}$.

Conjecture

$\boldsymbol{c} \in M$ and π satisfies the strong Slater condition, then Λ^* is bounded.

Lema

If $\sum_{t \in T} \lambda_t \boldsymbol{a}_t = \boldsymbol{c}$, with $\lambda_t \geq 0$ for all $t \in T$, then there exist $\gamma_i \geq 0$, $i = 1, \ldots, n+1$, such that,

$$\sum_{i=1}^{n+1} \gamma_i \boldsymbol{a}_{t_i} = \boldsymbol{c} \quad \text{and} \quad \sum_{t \in T} \lambda_t = \sum_{i=1}^{n+1} \gamma_i .$$

Theorem

Let $\pi = (\boldsymbol{a}, b, \boldsymbol{c})$ a parameter with $|T| \geq n+2$. If $\boldsymbol{c} \in M$ and π satisfies the strong Slater condition, then Λ^* is bounded with the norm $||\cdot||_1$.

Corollary

Let $\pi = (\boldsymbol{a}, b, \boldsymbol{c})$ a parameter with $|T| \geq n+2$. The following statements are true:

(i) If $\boldsymbol{c} \in int\ M$ where $\boldsymbol{c} = \boldsymbol{0}_n$ and π satisfies the strong Slater condition, then $\Lambda^* = \{\lambda \equiv 0\}$.

(ii) If $\boldsymbol{c} \in M$ where $\boldsymbol{c} \neq \boldsymbol{0}_n$ and π satisfies the strong Slater condition, then

$$\text{ínf } \{||\lambda||_1 \ : \ \lambda \in \Lambda^*\} > 0.$$

Corollary

Let $\pi = (\boldsymbol{a}, b, \boldsymbol{c})$ a parameter with $|T| \geq n + 2$. If $\boldsymbol{c} \in int\ M$ where $\boldsymbol{c} = \boldsymbol{0}_n$ and π satisfies the strong Slater condition, then $\pi \in \Pi_1^1$.

Corollary

Let $\pi = (\boldsymbol{a}, b, \boldsymbol{c})$ a parameter with $|T| \geq n+2$. If $\boldsymbol{c} \in int\ M$ where $\boldsymbol{c} = \boldsymbol{0}_n$ and π satisfies the strong Slater condition, then $\pi \in \Pi_1^1$.

The feasible set of the system $\{\boldsymbol{a}_t \boldsymbol{x} \geq b_t,\ t \in T\}$, (with $|T| \geq n+2$) is nonempty and bounded, if $\boldsymbol{0}_n \in int\ cone\ \{\boldsymbol{a}_t,\ t \in T\}$ and there are $\epsilon > 0$ and $\overline{\boldsymbol{x}} \in \mathbb{R}^n$ such that, $\boldsymbol{a}_t \overline{\boldsymbol{x}} > b_t + \epsilon$ for all $t \in T$.

Corollary

Let $\pi = (\boldsymbol{a}, b, \boldsymbol{c})$ a parameter with $|T| \geq n+2$. If $\boldsymbol{c} \in int\ M$ where $\boldsymbol{c} = \boldsymbol{0}_n$ and π satisfies the strong Slater condition, then $\pi \in \Pi_1^1$.

The feasible set of the system $\{\boldsymbol{a}_t \boldsymbol{x} \geq b_t,\ t \in T\}$, (with $|T| \geq n+2$) is nonempty and bounded, if $\boldsymbol{0}_n \in int\ cone\ \{\boldsymbol{a}_t,\ t \in T\}$ and there are $\epsilon > 0$ and $\overline{\boldsymbol{x}} \in \mathbb{R}^n$ such that, $\boldsymbol{a}_t \overline{\boldsymbol{x}} > b_t + \epsilon$ for all $t \in T$.

In fact, if the conditions are true and we consider the parameter $\pi = (\boldsymbol{a}, b, \boldsymbol{0}_n)$, then $\pi \in \Pi_1^1$. In particular, F^* is bounded and nonempty. As $F = F^*$ the result is immediate.

Corollary

Let $\pi = (\boldsymbol{a}, b, \boldsymbol{c})$ a parameter with $|T| \geq n+2$. If $\boldsymbol{c} \in M$ where $b \equiv 0$ and π satisfies the strong Slater condition, then $\pi \in \Pi_1^1$.

$D \setminus P$		IC	B		UB
			S	N	
IC		Π_4		Π_5	Π_2
B	S	Π_6^1	Π_1^1	Π_1^3	
	N	Π_6^2	Π_1^2	Π_1^4	
UB		Π_3			

Example 2. Consider in $\mathbb{R}^2$ the problem:

$$P_2 : \qquad \text{ínf } \; 0$$

$$s.t. \;\; tx_1 + tx_2 \geq 1, \qquad t \in (0,1].$$

The problem is inconsistent, because,

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = cl \; cone \left\{ \begin{pmatrix} t \\ t \\ 1 \end{pmatrix}, t \in (0,1] \right\}.$$

The dual problem is:

$$\begin{aligned} D_2: \qquad & \sup \sum_{t\in(0,1]} \lambda_t \\ s.t. \quad & \sum_{t\in(0,1]} \lambda_t \binom{t}{t} = \binom{0}{0} \\ & \lambda \in \mathbb{R}_+^{(T)}. \end{aligned}$$

We have

$$\sum_{t\in(0,1]} \lambda_t t = 0.$$

as $t \in (0,1]$, then

$$\sum_{t\in(0,1]} \lambda_t = 0.$$

Therefore

$$v^D(\pi^2) = 0 \text{ y } \Lambda_2^* = \{\lambda \equiv \overline{0}\}.$$

Conclusion

We conclude mentioning that we have obtained a sufficient condition for the boundedness of the optimal set of the dual problem, which might however be empty. Conditions that guarantee the solvability of the dual problem turns out to be complicated task even in the continuous case. This could be a challenge problem for a future work.

Open problem

The characterization of the solvable dual problems with bounded optimal set

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Thank you