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Portfolio optimization with incorporation of preferences and many criteria

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Graphical Abstract	**Abstract.**
Insert grafical abstract figure here	Portfolio optimization is one of the most addressed areas in operational research, mainly because of its practical relevance and interesting theoretical challenges. Recently, Solares *et al.* (2018) have proposed using probabilistic confidence intervals as criteria to select the most convenient portfolio. An approach following this idea allows the investor to consider not only the expected impact of the portfolios but also the risk of not obtaining that expected impact. Moreover, it identifies the behavior of the investor in presence of risk and gives her/him support depending on her/his own preferences. On the other hand, there are situations where the investor is not satisfied with the knowledge provided by probabilistic information (e.g., such information is precarious or the investor gives importance to other information, such as financial data). In this case, the investor may be interested in considering many criteria in order to select the most convenient portfolio.

However, this is not a trivial task since the cognitive limitations make it very difficult for the investor to consistently select the best compromise in presence of many criteria. Bearing this in mind, Fernandez *et al*. (2018a) proposed an approach that aggregates the many criteria on the basis of the investor's particular system of preferences producing a selective pressure towards the most preferred portfolio while the investor's cognitive effort in the final selection is reduced.

1. Introduction

A problem faced by most organizations and individual investors is how to distribute a monetary amount among a set of investment objects in such a way as to maximize the impact on their objectives. The process of allocating resources maximizing the impact on the investor's (decision maker, DM) objectives is known as Portfolio Optimization.

Since the 1970s we have seen an accelerated evolution in several fields of science, such as finance, optimization and decision making. It is due to the evolution in these fields that various researchers have made significant advances in Information Theory, generating new products and financial services. However, there is still a collection of challenges that leads to an increasingly large number of papers that consider

- multiple conflicting objectives,
- analysis of the investment objects' performances,
- selection of the best investment objects,
- risk management,
- the specific risk behavior of the decision maker, and
- the particular preferences of the decision maker.

From all these objectives, the most outstanding one is maximization of the portfolio's return/profit (Solares et *al*, 2018). The return of a given portfolio is the arithmetic difference between the buy-cost and the sell-cost of the portfolio. The maximization of the portfolio's return is sometimes the only objective optimized during the allocation of resources; however, given the high complexity involved in the return's forecasting procedure, many criteria (e.g., expected return, risk, so-called fundamental and technical analyses) usually underlie such objective. Here, we will describe two papers that address the latter situation; namely, Refs. (Solares et *al*, 2018) and (Fernandez et *al*, 2018a).

Note that depending on the context, for example if we just want to maximize the *expected* return, the Portfolio Optimization problem is easy to solve. We simply put as much money as possible into the highest (expected) returning investment object. The reason that there is much work on this subject is that there is a requirement to control risk, which is better achieved when supporting several investment objects. Therefore, distributing a monetary amount among a set of investment objects in order to maximize the impact on the investor's objectives requires the estimation of the *joint* impact produced by the supported investment objects, as well as the *joint* risk of not achieving such impact. This is due to the concept of risk diversification. It is traditionally based on the idea that a portfolio's (set of supported investment objects) riskiness depends on the correlation of its constituents, not only on the average riskiness of its separate holdings. Even when diversification is not too relevant in some scenarios (e.g., when the investment objects behave like certain so-called stable Paretian distributions; Fama, 1965), generally, most practitioners agree that a certain level of diversification is achievable (Fabozzi, 2007c). Thus, the idea of evaluating the distribution of resources in terms of portfolios is opposed to the belief that investors should invest in the (individual) investment objects that offer the highest future impact. Furthermore, we assume here that investment objects are correctly assessed and concentrate on how to select the best portfolio in terms of the investor's preferences. Finally, it is important to highlight that even in presence of the same level of risk, two decision makers with different attitudes facing risk might have different levels of satisfaction. Hence, besides a risk measure, the approaches finding the best portfolios must incorporate the DM's attitude in presence of risk during the Portfolio Optimization.

Investors frequently use decision-aiding tools in order to obtain a set of portfolios representing, to a certain extent, the best feasible allocations of resources. But this does not solve the problem; the investor still must choose from among all these portfolios the one that represents the best compromise among the considered criteria. But, as reported by Miller (1956), this is not a trivial task since the cognitive limitations make it very difficult for the investor to consistently select the best compromise in the presence of many criteria. This becomes more complicated when she/he needs to make trade-offs between risk and return. Consequently, a more convenient approach must be followed; the goal of such an approach must be to provide a minimal set of portfolios satisfying the investor's preferences.

The main objectives in Refs. (Solares et *al*, 2018) and (Fernandez et *al*, 2018a) are to present approaches capable to find the most satisfactory portfolio from the investor's perspective when many objective functions are considered. In those works, the investor's behavior facing risk, the estimations of the portfolios' future returns, and the risk of not attaining those returns are all represented through probabilistic confidence intervals. The imperfect knowledge related to the subjectivity of the investor is modeled on the basis of Interval Theory and the interval-based outranking, allowing one to obtain an approximation to the investor's preferences. These preferences are used by the authors' approach to perform an aggregation of all the criteria. Thus, a selective pressure towards the most preferred portfolio is produced, while the investor's cognitive effort in the final selection is reduced.

2. Materials and Methods

2.1 Portfolio Optimization

In this section, we formalize the definition of the Portfolio Optimization problem and, without loss of generality, we concentrate in the rest of the document in portfolios composed by stocks, a type of asset that represents partial ownership of a corporation. Let us assume that there are n stocks (investment objects) where the investor can allocate a proportion of his/her resources. Let $p_0^i, p_1^i, p_2^i, \cdots, p_{T_0}^i$ be the historical prices of the ith stock in $T_0 + 1$ periods of time. The T_0 historical rate of returns (only *returns* in the following) of the stock are given by $r_t^i = (p_t^i - p_{t-1}^i)/p_{t-1}^i;\ t = 1, 2, \cdots, T_0$. Here, a portfolio is a vector $x = [x_1, x_2, \ldots, x_n]^\top$ in decision space that specifies the proportions of money to invest in n investment objects, such that x_i is the proportion to invest in the ith object. The image of a portfolio in objective space is a real number that states the return produced by the portfolio, $R(x)$. It is widely accepted in the literature that the return produced by the portfolio x can be obtained as (cf. Fabozzi *et al*., 2007):

$$R(x) = \sum_{i=1}^{n} x_i r_{T_0+1}^i.$$

The issue that investors (Decision Makers) face with the previous definition is that it depends on the future returns of the stocks, $r_{T_0+1}^i$, whose values are unknown. Thus, underlying criteria are used with the purpose of estimating such returns, creating the criteria space. The image of a portfolio in the criteria space is a vector that represents the impact of the portfolio on k criteria. The portfolio problem is then to select the feasible portfolio that maximizes the impact on the criteria. Formally:

$$\underset{x \in \Omega}{\text{maximize}}\left(I(x) = \{I_{1(x)}, I_{2(x)}, \cdots, I_{k(x)}\}\right), \tag{1}$$

where $I_j(x)$ is the impact of portfolio x on criterion j and Ω is the set of feasible portfolios (the set of portfolios that fulfill the decision maker's constraints).

2.2 Interval Theory

The so-called Interval Theory was originated independently by Sunaga (1958) and Moore (1962). Interval Theory's principal concept is the *interval number*. Such a number represents a numerical quantity whose exact value is unknown. Given this imperfect knowledge about the quantity, a range of numbers is used to encompass all the possible values that the quantity could obtain. In this way, an interval number stands for an indeterminate number that takes its possible value within a set of numbers. Let us consider the quantity ι whose real value lies between bounds i^- and i^+. The interval number for such quantity is set then as $I = [i^-, i^+]$. We can also translate a real number, q, into an interval number as $[q, q]$.

In what follows, let us look at the basic operations of interval numbers. Given the interval numbers $I = [i^-, i^+]$ and $J = [j^-, j^+]$, the following equations represent the addition, subtraction, multiplication and division, of I and J, respectively.

$$I + J = [i^- + j^-, i^+ + j^+],$$
$$I - J = [i^- - j^+, i^+ - j^-],$$
$$I \times J = [\min\{i^- j^-, i^- j^+, i^+ j^-, i^+ j^+\}, \max\{i^- j^-, i^- j^+, i^+ j^-, i^+ j^+\}],$$
$$I \div J = [i^-, i^+] \times \left[\frac{1}{j^-}, \frac{1}{j^+}\right].$$

More recently, Shi *et al.* (2005) proposed a way to determine the order of interval numbers. For instance, suppose we want to determine the order of $I = [i^-, i^+]$ and $J = [j^-, j^+]$. First, we need to find the possibility of I being greater than or equal to J. The possibility function proposed in (Shi *et al.*, 2005) is given by

$$p(I \geq J) = \begin{cases} 1 & \text{if } p_{\{IJ\}} > 1, \\ p_{\{IJ\}} & \text{if } 0 \leq p_{\{IJ\}} \leq 1, \\ 0 & \text{if } p_{\{IJ\}} < 0. \end{cases} \quad (2)$$

Where $p_{\{IJ\}} = \frac{i^+ - j^-}{(i^+ - i^-) + (j^+ - j^-)}$.

Furthermore, if $i^+ = i^-$ and $j^+ = j^-$, then

$$p(I \geq J) = \begin{cases} 1 & \text{if } I \geq J, \\ 0 & \text{otherwise.} \end{cases}$$

Let i and j be two currently undetermined realizations from I and J, respectively; $p(I \geq J)$ can be interpreted as a degree of credibility of the statement "once both realizations are determined, i will be greater than or equal to j". This helps the DM to ensure the robustness of $I \geq J$, that is, to have a strong belief on I being not less than J when they are instanced as real numbers (Fernandez *et al.*, 2018b).

2.3 Handling uncertainty through confidence intervals in Portfolio Optimization

Let $R(x)$ be a random variable that represents the return of portfolio x and $\mathbb{P}(\omega)$ the probability that event ω will occur. Then, $\theta_\gamma(x) = [\alpha, \beta]: \mathbb{P}(\alpha \leq R(x) \leq \beta) = \gamma$ is called confidence interval of the portfolio return; it is possible to consider multiple confidence intervals $\theta_{\gamma_j}(x)$. Furthermore, each γ_j is selected by the investor according to his/her own preferences. This allows one to incorporate his/her attitude facing risk in the following manner. First, suppose a highly risk-averse investor; such investor would feel more satisfied of making a decision based on intervals with a high probability of containing the actual return. That is, he/she considers more valuable the information about the worst scenarios that could happen when selecting a portfolio; thus, he/she would select high values for γ_j looking for protection against those scenarios (see Figure 1(a) of Ref. (Solares et *al*., 2018)). On the other hand, if the investor is lowly risk-averse, he/she would prefer to make a decision based on intervals that tend to the expected return (see Figure 1(b) of Ref. (Solares et *al*., 2018)).

Solares et *al*. (2018) propose to characterize the portfolios using confidence intervals as criteria underlying risky objectives. Hence, the best portfolio is the one that maximizes the set of confidence intervals whose probabilities of containing the portfolio return are given by the investor. It is important to note that the maximization referred to in that work is not necessarily related to the wideness of the intervals, but it is based on the possibility function defined in Equation (2). That is, portfolios with the rightmost confidence intervals are preferred.

Each $\theta_{\gamma_j}(x)$ is easily understandable even for an investor without a sophisticated technical preparation, since it represents the probability that the return of portfolio x actually lies within the interval $[\alpha_j, \beta_j]$. This is not the case if one considers the technical criteria used in the mean-variance approach (Markowitz, 1952) or higher statistical moments (Saranya, 2014; Scott, 1980; Dittmar,2002; Harvey, 2000).

Moreover, the investor has the capability of defining as many criteria per objective as (s)he wishes; thus, the information describing the distribution is enough to satisfy his/her requirements. Nevertheless, we believe that no more than one or two criteria are sufficient to satisfy his/her requirements for information. This is because of the definition of each $\theta_{\gamma_j}(x)$, which allows the approach to encompass multiple points of the probability distribution in a single criterion. That is, in a single criterion we know with a given probability that the portfolio's return can be *any* of the values within the corresponding interval. This is not possible in point estimators, where the statistical information relies on only one point. In some approaches (see e.g., Greco *et al*., 2013; Markowitz, 1952; Markowitz, 1968) each criterion represents a single point of the probability distribution, so a better description of the distribution requires a higher number of criteria.

2.4 Considering the decision maker's preferences to select the best portfolio in presence of many criteria

As stated in the Introduction, there are situations addressing the Portfolio Optimization problem where many criteria are used to characterize the decision alternatives. Once these criteria are stated, the decision maker has the possibility to directly address the multi-criteria problem trying to arrive to its Pareto frontier. Of course, this does not solve the problem; rather, it is useful to find equivalent alternative solutions that are not *inferior,* the decision maker still needs to select what she/he considers the portfolio with the best compromise among the impacts on the criteria. Another option for the optimization procedure is to incorporate the decision maker's preferences during the search, thus creating a pressure towards the decision maker's region of interest within the Pareto frontier and, eventually, selecting the most preferred portfolio.

The proposal presented in Ref. (Fernandez *et al*., 2018a) exploits the so-called interval-based outranking approach (cf. Fernandez *et al*., 2018b), a generalization of the outranking approach, in order to reproduce the decision maker's system of preferences during the search of the most preferred portfolio. As in the classical version of the outranking approach, the one based on Interval Theory uses a set of numerical parameters to model the DM's preferences. In addition to the former, the latter is able to denote the uncertainty involved in the decision maker's system of preferences (cf. Roy, 2014 to see a discussion about this uncertainty) by assuming that some preference parameters are defined as interval numbers. The set of parameters, $\mathcal{P} = \{w_j, v_j, \lambda, \beta_0\}, j = 1, \cdots, k$, used by the interval-based outranking approach is composed of a weight and a veto threshold for each criterion and two cutting levels that allow to determine if a crisp outranking relation between two decision alternatives holds. Following Refs. (Fernandez et *al*., 2010 and 2011), Fernandez et *al*. (2018a) use the interval-based outranking approach and a specification of Fuzzy Logic, the so-called Compensatory Fuzzy Logic based on the geometric mean (Espin *et al*., 2014), to define the *non-outranked truth degree* of portfolio x.

A high non-outranked truth degree indicates the lack of arguments to believe that there are better solutions than x in a given set of portfolios. Thus, by maximizing it, the optimization procedure is actually increasing the decision maker's satisfaction in the selected portfolio.

On the other hand, a high non-outranked degree is a necessary condition to be the best compromise, but it is not sufficient. A solution may have a high non-outranked degree and be incomparable with all or many of the solutions in the known Pareto front. Positive arguments are required to affirm the superiority of x over the other optimal solutions under consideration when they have the same non-outranked degree. Thus, Fernandez et *al*. (2018a) propose to use the outranking net flow score (Fodor and Roubens, 1994), as a "tie-breaker measure" when two portfolios have the same non-outranked truth degree.

Results and Conclusions

Both Refs. (Solares *et* al., 2018) and (Fernandez et *al*., 2018a) assess the performances of their respective proposed approaches by simulating risk-averse investors interested in many criteria and using real historical data in the context of stock Portfolio Optimization. They contemplate the most turbulent and uncertain period of the stock-market in the last years within their experimentation, the crisis of years 2007-2009. Sufficient evidence was found there to argue that i) using confidence intervals to characterize portfolios was crucial for the proposed approaches to select the portfolios according to the investor's behavior when facing risk, being conservative when such behavior is highly risk averse and taking good advantage of the uptrends when the behavior is lowly risk averse; ii) an optimization based on evolutionary algorithms and Interval Theory allowed the proposed approaches to outperform several important and well-established benchmarks; iii) Fernandez et *al*. (2018a) showed that their proposed approach is capable to provide the investor with portfolios that, generally, satisfy more her/his preferences than other benchmarks, even when these preferences are ill-determined, imprecise, uncertain or arbitrary.

The approach presented in (Fernandez et al., 2018a) is interesting since it looks for selecting the best portfolio considering the imperfect knowledge (in the sense of Roy et al., 2014) that characterizes the DM's implicit model of preferences, performing a selective pressure towards the DM's region of interest within the Pareto front; it allows to represent the DM's conservatism in the presence of risk, the portfolios' most probable returns and the risk of not attaining those returns. However, a direct elicitation of the parameter values representing the DM's preferences is performed in (Fernandez et al., 2018a). As raised by Mousseau and Slowinski (1998), it is often difficult for the DM to express specific values for the parameters of models representing his/her own preferences; this is true even when the parameters are defined as ranges of numbers.

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