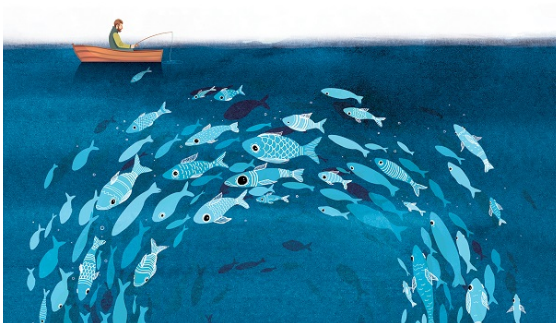
MOL2NET, International Conference Series on Multidisciplinary Sciences
http://sciforum.net/conference/mol2net-03

Mathematical modeling of a case of sustainable exploitation of a biological species

J. L. González-Santander (E-mail: juanluis.gonzalezsantander@gmail.com) [a],
G. Martín González (E-mail: german.martin@ucv.es) [a].
[a]*Universidad Católica de Valencia "san Vicente mártir"*

Graphical Abstract



Abstract
We pose the problem of the introduction of a biological species in an ecosystem in order to a future exploitation. For the first vivarium stage, we have obtained a simple method to determine the parameters of the logistic equation, which determines the growth of the species, from two experimental measures of the number of individuals. For the second phase of exploitation, we have obtained an expression for the maximum exploitation rate (number of individuals caught per unit time) for it to be sustainable, that is, the population does not become extinct. In the event that the exploitation rate is higher than the maximum that ensures sustainability, we provide an expression for the time of extinction. It has also obtained an expression for the minimum time that must have the vivarium stage to maximize sustainable harvest rate.

Results and discussion

For the vivarium stage (i.e. $0 \le t \le t_p$), we consider that the population of the species follows the logistic equation.

$$N(t) = \frac{N_\infty}{1 - Ce^{-kt}},$$

If $N(0) = N_0$, $N(t_m) = N_1$ and $N(2t_m) = N_2$, where $2t_m < t_p$, then

$$C = \frac{N_2 (N_0 - N_1)^2}{N_0 (N_2 N_0 - N_1^2)},$$

$$k = \frac{1}{t_m} \log\left[\frac{N_2 (N_0 - N_1)}{N_0 (N_1 - N_2)}\right],$$

$$N_\infty = \frac{N_1 \left[N_0 (N_1 - 2N_2) + N_2 N_1\right]}{N_1^2 - N_2 N_0}.$$

For the exploitation stage, we consider that the population is governed by the following ODE,

$$\frac{dN}{d\tau} = kN\left(1 - \frac{N}{N_\infty}\right) - m, \qquad N(0) = N_0^* > 0, \qquad \tau = t - t_p \ge 0,$$

where m denotes the number of individuals caught per unit time. The solution to the above ODE reads as follows:

1) *Sigmoidal solution* ($m < k N_\infty/4$)

$$N(\tau) = N_- + \frac{N_+ - N_-}{1 - K\exp(-\Delta\tau)},$$

where

$$\begin{cases} N_\pm = \frac{1}{2}\left(N_\infty \pm \sqrt{N_\infty\left(N_\infty - 4\frac{m}{k}\right)}\right), \\ \Delta = \sqrt{k\left(k - \frac{4m}{N_\infty}\right)} > 0, \quad K = \frac{N_+ - N_0^*}{N_- - N_0^*}. \end{cases}$$

In this case, the species gets extinct when $N_0^* < N_\infty/2$ or when $m > kN_0^*(1 - N_0^*/N_\infty)$ at time

$$\tau_0 = -\frac{1}{\Delta}\log\left[\frac{\left(N_0^* - N_-\right)N_+}{\left(N_0^* - N_+\right)N_-}\right].$$

2) *Trigonometric solution* ($m > k N_\infty/4$)

$$N(\tau) = \delta - \gamma\tan(\alpha\tau + \beta),$$

where

$$\begin{cases} \alpha = \frac{1}{2}\sqrt{k\left(\frac{4m}{N_\infty} - k\right)}, \quad \tan\beta = \frac{\delta - N_0^*}{\gamma}, \\ \gamma = \frac{1}{2}\sqrt{\left(\frac{4m}{k} - N_\infty\right)N_\infty}, \quad \delta = \frac{1}{2}N_\infty. \end{cases}$$

In this case, the population always gets extinct at time

$$\tau_0 = \frac{1}{\alpha}\left[\tan^{-1}\left(\frac{\delta}{\gamma}\right) - \tan^{-1}\left(\frac{\delta - N_0^*}{\gamma}\right)\right].$$

3) *Singular solution* ($m = k N_\infty/4$)

$$N(\tau) = \frac{N_\infty}{2} + \frac{\left(2N_0^* - N_\infty\right)N_\infty}{k\left(2N_0^* - N_\infty\right)\tau + 2N_\infty}.$$

In this case, the population gets extinct when $N_0^* < N_\infty/2$ at time

$$\tau_0 = \frac{4N_0^*}{k\left(N_\infty - 2N_0^*\right)}.$$

According to the above solution, the maximum exploitation rate for which the species does not get extinct is given by

$$m_{\max} = \begin{cases} kN_0^*\left(1 - \frac{N_0^*}{N_\infty}\right), & N_0^* \le \frac{N_\infty}{2}, \\ \frac{kN_\infty}{4}, & N_0^* > \frac{N_\infty}{2}. \end{cases}$$

Defining t_v as the minimum time we must wait to start the exploitation in order to obtain $m_{\max}$ being sustainable, we have

$$t_v = \begin{cases} \dfrac{1}{k}\log\left(\dfrac{N_\infty}{N_0} - 1\right), & N_0 < \dfrac{N_\infty}{2}, \\ 0, & N_0 \geq \dfrac{N_\infty}{2}. \end{cases}$$

Therefore, that the second stage must start at $t_p \geq \max(t_v, 2t_m)$.

References

(1) Verhulst, Pierre-François (1838), “Notice sur la loi que la population poursuit dans son accroissement”, *Correspondance mathématique et physique* 10: 113–121.

(2) Malthus, Thomas Robert (1798), *An Essay on the Principle of Population*, Library of Economics, Liberty Fund, Inc., 2000.

(3) Weisstein, Eric W. "Sigmoid Function." From MathWorld--A Wolfram Web Resource. http://mathworld.wolfram.com/SigmoidFunction.html

(4) Spiegel, Murray R., *Manual de fórmulas y tablas matemáticas*, McGraw-Hill, 1982.

(5) http://en.wikipedia.org/wiki/Inequality_(mathematics)