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Information length as a new diagnostic of stochastic resonance

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Abstract:
Stochastic resonance is a subtle, yet powerful phenomenon in which a noise plays an interesting role of amplifying a signal instead of attenuating it. Popular measures to study stochastic resonance include signal-to-noise ratios, residence time distributions, and different information theoretic measures. Here, we show that the information length provides a novel method to capture stochastic resonance -- the information length measures the total number of statistically different states along the path of a system. Specifically, we consider the classical double-well model of stochastic resonance, in which a particle in a potential $V(x, t) = [-x^2/2 + x^4/4 - A \sin(\omega t)\, x]$ is subject to an additional stochastic forcing. We present direct numerical solutions of the Fokker-Planck equation for the probability density function $p(x, t)$, for $\omega = 10^{-2}$ to 10^{-6}, and $A \in [0,0.2]$ and show that the information length shows a very clear signal of the resonance. That is, stochastic resonance is reflected in the total number of different statistical states that a system passes through.

Key words: Stochastic resonance; Fokker-Planck equation; Probability density function; Information geometry; Information length

Outline

1. Introduction on information length
2. Double-well potential model
3. Results
4. Conclusion

1. Introduction on information length [1-11]

τ=characteristic timescale of information change : $dL = \frac{dt}{\tau}$

$$\frac{1}{\tau^2(t)} = \int dx\; p(x,t)\left(\frac{\partial \ln p(x,t)}{\partial t}\right)^2 = \left(\frac{dL}{dt}\right)^2 \qquad (1)$$

L(t): total number of statistically different states that a system undergoes in time *(0,t)*

$$L(t) = \int_0^t dL = \int_0^t \frac{dt_1}{\tau(t_1)} \qquad (2)$$

Relation to infinitesimal relative entropy [10]

We consider two nearby PDFs $p_1 = p(x,t_1)$ and $p_2 = p(x,t_2)$ at time $t = t_1$ and t_2 and the limit of a very small $\delta t = t_2 - t_1$ to do Taylor expansion of $D[p_1,p_2] = \int dx\, p_2 \ln(p_2/p_1)$ by using

$$\frac{\partial}{\partial t_1} D[p_1,p_2] = -\int dx\, p_2 \frac{\partial_{t_1} p_1}{p_1}, \tag{3}$$

$$\frac{\partial^2}{\partial t_1^2} D[p_1,p_2] = \int dx\, p_2 \left\{ \frac{(\partial_{t_1} p_1)^2}{p_1^2} - \frac{\partial_{t_1}^2 p_1}{p_1} \right\}, \tag{4}$$

$$\frac{\partial}{\partial t_2} D[p_1,p_2] = \int dx \left\{ \partial_{t_2} p_2 + \partial_{t_2} p_2 [\ln p_2 - \ln p_1] \right\}, \tag{5}$$

$$\frac{\partial^2}{\partial t_2^2} D[p_1,p_2] = \int dx \left\{ \partial_{t_2}^2 p_2 + \frac{(\partial_{t_2} p_2)^2}{p_2} + \partial_{t_2}^2 p_2 [\ln p_2 - \ln p_1] \right\}. \tag{6}$$

In the limit $t_2 \to t_1 = t$ ($p_2 \to p_1 = p$), Equations (3)–(6) give us

$$\lim_{t_2 \to t_1} \frac{\partial}{\partial t_1} D[p_1,p_2] = \lim_{t_2 \to t_1} \frac{\partial}{\partial t_2} D[p_1,p_2] = \int dx \partial_t p = 0,$$

$$\lim_{t_2 \to t_1} \frac{\partial^2}{\partial t_1^2} D[p_1,p_2] = \lim_{t_2 \to t_1} \frac{\partial^2}{\partial t_2^2} D[p_1,p_2] = \int dx \frac{(\partial_t p)^2}{p} = \frac{1}{\tau^2}$$

Relation to infinitesimal relative entropy - continued

Up to $O((dt)^2)$ $(dt = t_2 - t_1)$

$$D[p_1, p_2] = \frac{1}{2}\left[\int dx \frac{(\partial_t p(x,t))^2}{p(x,t)}\right](dt)^2$$

and thus the infinitesimal distance $dl(t_1)$ between t_1 and $t_1 + dt$ as

$$dl(t_1) = \sqrt{D[p_1, p_2]} = \frac{1}{\sqrt{2}}\sqrt{\int dx \frac{(\partial_{t_1} p(x,t_1))^2}{p(x,t_1)}}dt.$$

By summing $dt(t_i)$ for $i = 0, 1, 2, \ldots, n-1$ (where $n = t/dt$) in the limit $dt \to 0$, we have

$$\lim_{dt\to 0}\sum_{i=0}^{n-1} dl(idt) = \lim_{dt\to 0}\sum_{i=0}^{n-1}\sqrt{D[p(x,idt), p(x,(i+1)]}\, dt \propto \int_0^t dt_1 \sqrt{\int dx \frac{(\partial_{t_1} p(x,t_1))^2}{p(x,t_1)}} = \mathcal{L}(t)$$

Merit of information length

- Path-dependent: when a probability density function continuously changes with time, the information length measures the total number of different statistical states that a system passes through in time [1-11]
- Enables us to quantify the difference in the dynamics between two time points
- Preserves a linear geometry of the Ornstein-Uhlenbeck process by a linear relation between the information length (in the long time limit) and the mean position of an initial Gaussian probability density function unlike relative entropy, Jenson divergence, etc (see sciforum-027580 by Heseltine & Kim)
- Captures a sensitive dependence on initial conditions in chaotic systems [1]
- Invariant under time-independent coordinate transformations unlike entropy

2. Double-well model [11]

Over-damped stochastic system with a periodic potential V(x,t)

$$\frac{dx}{dt} = -\frac{\partial}{\partial x} V + \xi = x - x^3 + A\sin(\omega t) + \xi$$

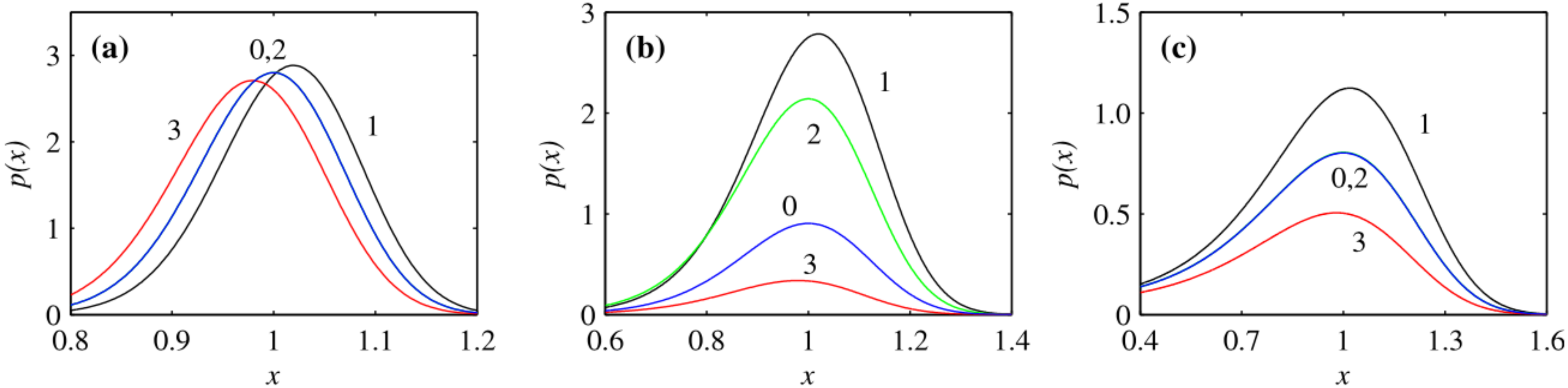
$$V(x,t) = -\frac{x^2}{2} + \frac{x^4}{4} - A\sin(\omega t)x$$

$$\langle \xi(t) \rangle = 0, \qquad \langle \xi(t_1)\xi(t_2) \rangle = 2D\delta(t_1 - t_2)$$

Fokker-Planck equation for probability density function p(x,t)

$$\frac{\partial}{\partial t} p(x,t) = -\frac{\partial}{\partial x}\Big((x - x^3 + A\sin(\omega t))\, p(x,t) \Big) + D\frac{\partial^2}{\partial x^2} p(x,t)$$

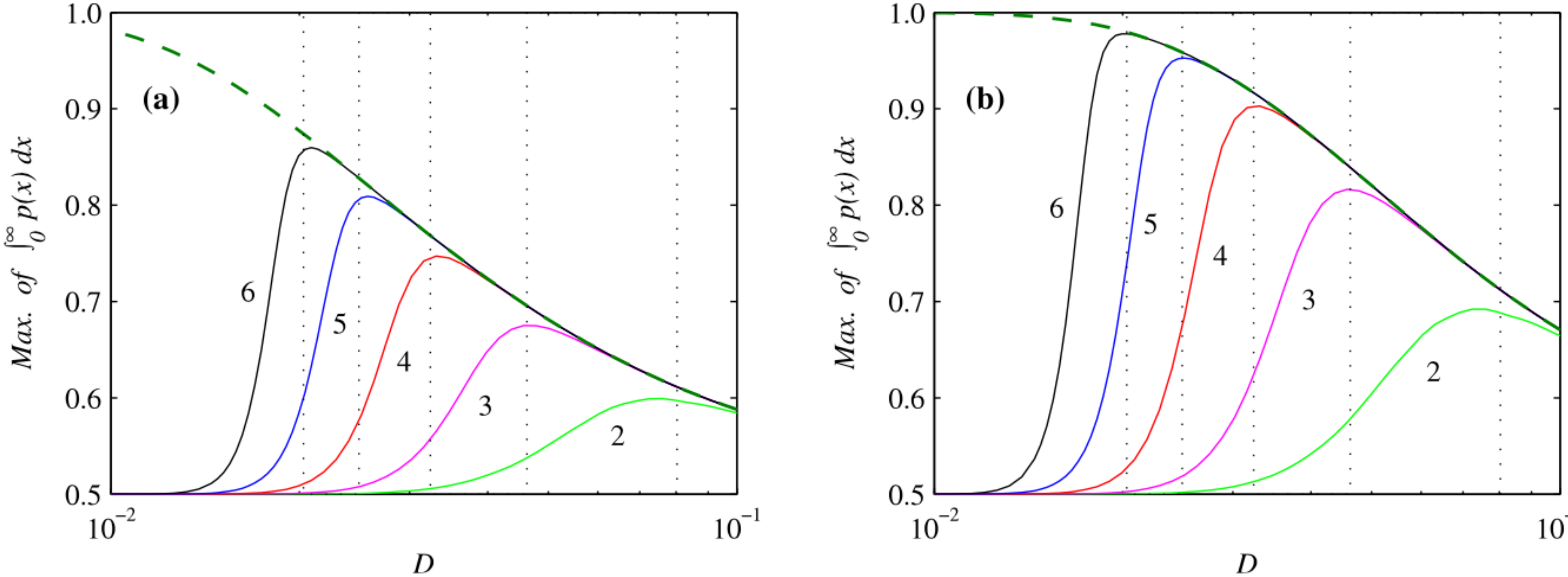
3. Results



The PDFs $p(x,t)$ at four times throughout the cycle, with the numbers $n = 0 - 3$ beside individual curves corresponding to $t = n\,T/4 \bmod (T)$. All three panels are for $\omega = 10^{-4}$ and $A = 0.04$, and (a) $D = 0.01$, (b) $D = 0.0324$, (c) $D = 0.1$.

Escape rate for the unperturbed system (A=0)

$$r_K = \frac{1}{\pi\sqrt{2}} \exp\left[-\frac{1}{4D}\right] \xrightarrow{\omega = r_K} D_{\mathrm{res}} = \frac{-1}{4\ln(\pi\sqrt{2}\,\omega)}$$



The maxima over the cycle of $\int_0^\infty p(x,t)\,dx$, as functions of the noise level D. The numbers 2 to 6 beside individual curves correspond to $\omega = 10^{-2}$ to 10^{-6}. (a) $A = 0.02$, (b) $A = 0.04$. The thick dashed curves show results from (10). The dotted vertical lines are at D_{res} given by (9) for $\omega = 10^{-2}$ to 10^{-6}; note how well these values agree with the maxima over D of the corresponding curves.

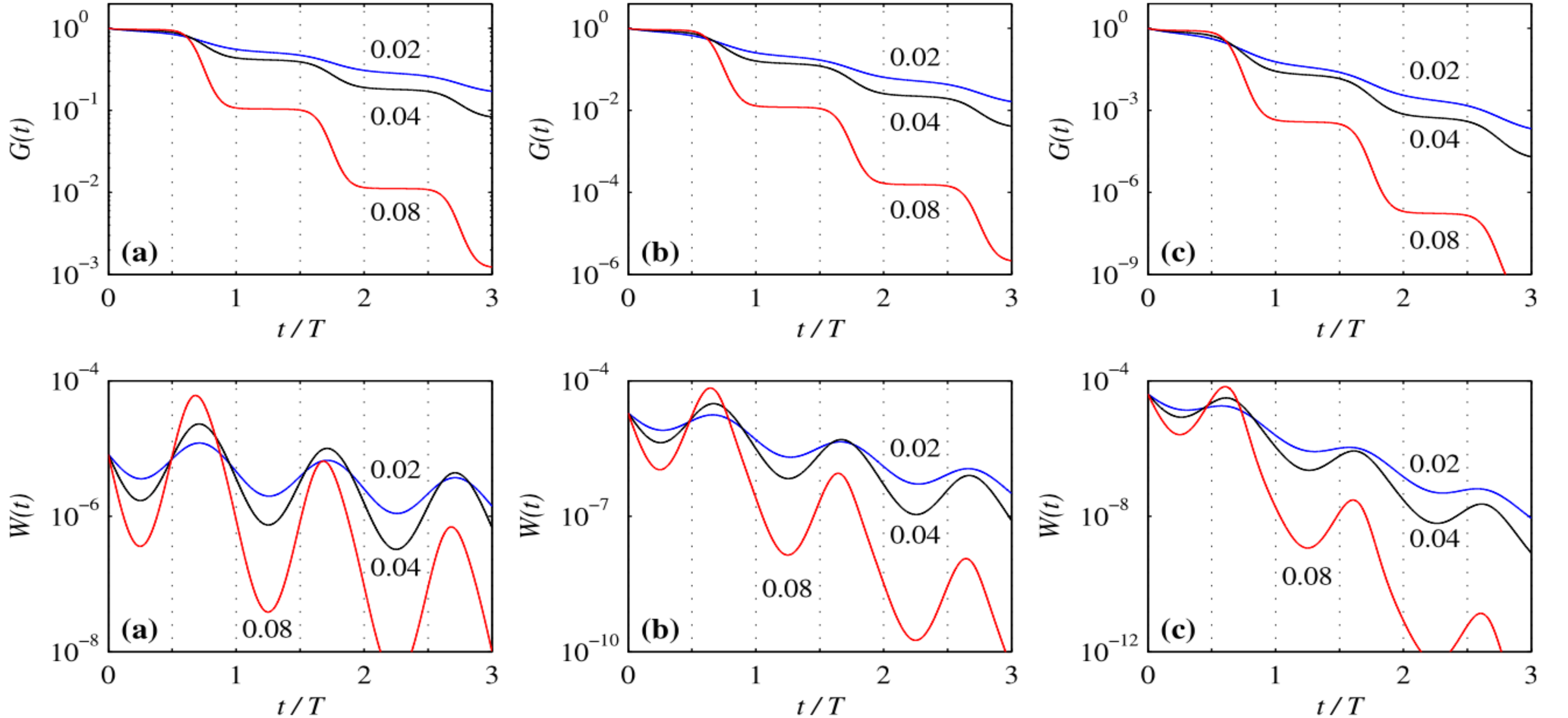
Summary of observations

First, if $D \ll D_{\text{res}}$, then $\omega \gg r_K$. The imposed modulation is then too rapid, the system cannot effectively respond, and the probability of being in either well remains essentially 0.5 throughout the entire cycle. That is, there is no synchronization between the stochastic switching and the modulation, and hence no resonance. In contrast, if $D \gg D_{\text{res}}$, then $\omega \ll r_K$. The imposed modulation is then so slow that the adiabatic limit (8) does indeed apply, even to a process as slow as the switching between the wells. And as the results from (10) or (11) show, the adiabatic formula (8) exhibits synchronization, and in particular stronger synchronization for smaller D, explaining why D should be as small as possible, but not much less than D_{res}, which would switch the resonance off.

Survival probability G(t) and Escape rate W(t)

A popular tool to study stochastic resonance is to investigate the escape of particles from a single well. That is, suppose we solve the same Fokker-Planck equation (7) as before, but now only on the interval $x \in [0,3]$. The boundary condition at $x = 0$ is $p = 0$, meaning that any particles that reach $x = 0$ are simply lost to the system. And indeed, the 'total probability' integral, $\int p(x,t)\,dx$, no longer remains constant in this formulation, but instead decreases in time, corresponding to the continual loss of particles at $x = 0$. The question then is, can we analyze and interpret this loss of particles in a similar way to the previous results, and in particular can we see the signature of resonance here as well?

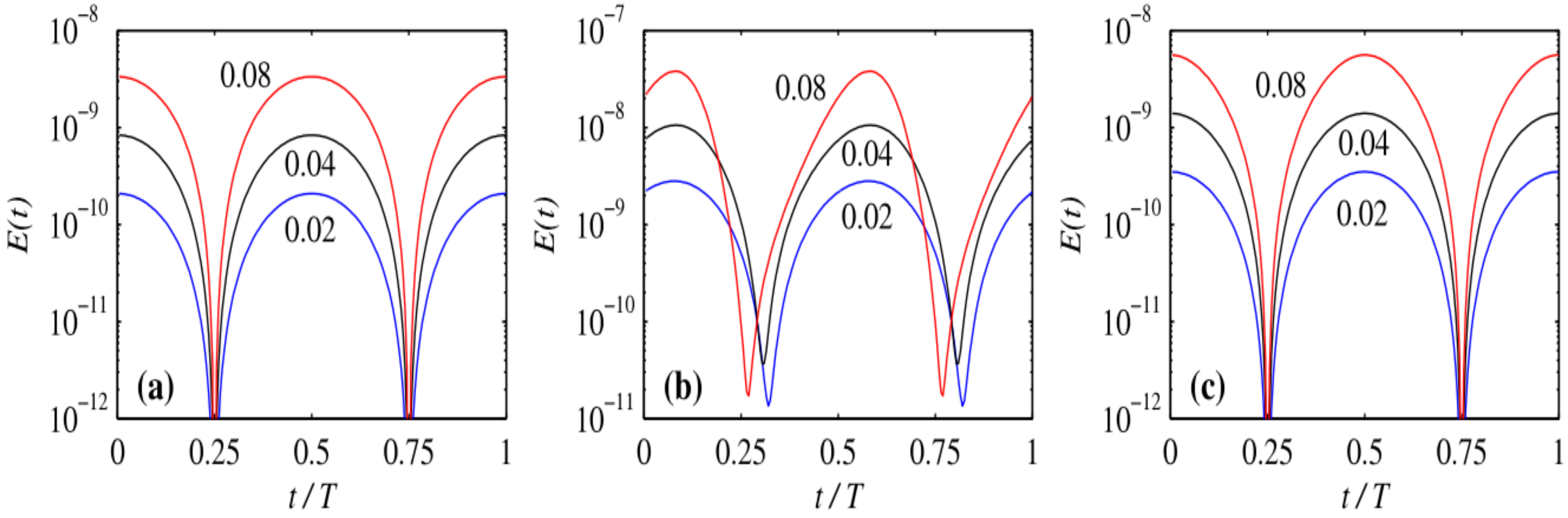
Starting from some suitable peak within the $x \approx 1$ well, integrating for just a few periods yields a solution that is again periodic in time, but now also decreases by a constant factor each period. That is, the solutions are of the form $p(x, t+T) = c\, p(x, t)$, where $c < 1$ is some factor that depends on the parameters D, ω and A, but is the same for each subsequent period once this behaviour has emerged. Also, because there is now only one well, the previous symmetrization procedure does not need to be applied, and this behaviour still arises after just a few periods.



The top row shows $G(t) = \int_0^\infty p(x,t)\,dx$, and the bottom row the corresponding $W(t) = -\frac{d}{dt}G(t)$. All three solutions are for $\omega = 10^{-4}$, (a) $D = 0.023$, (b) $D = 0.025$, (c) $D = 0.027$, and $A = 0.02$, 0.04 and 0.08 as indicated by the numbers beside individual curves.

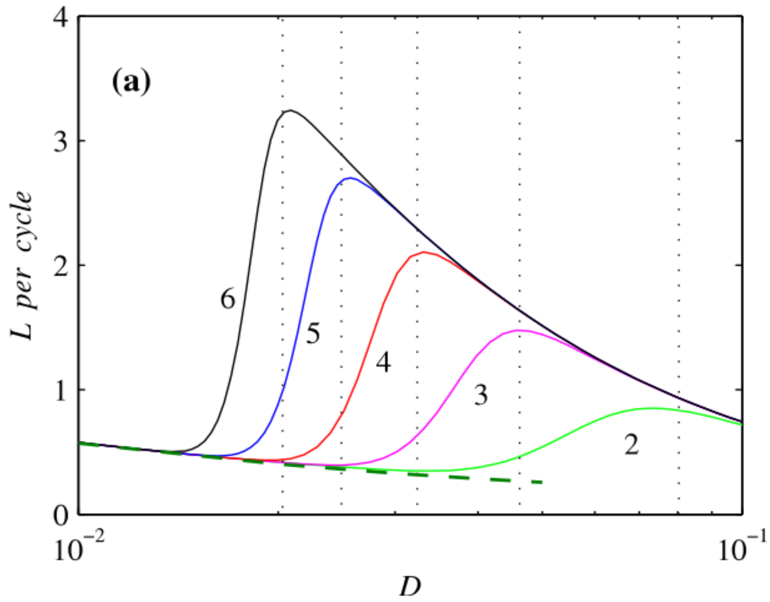
Information length *L(t)*

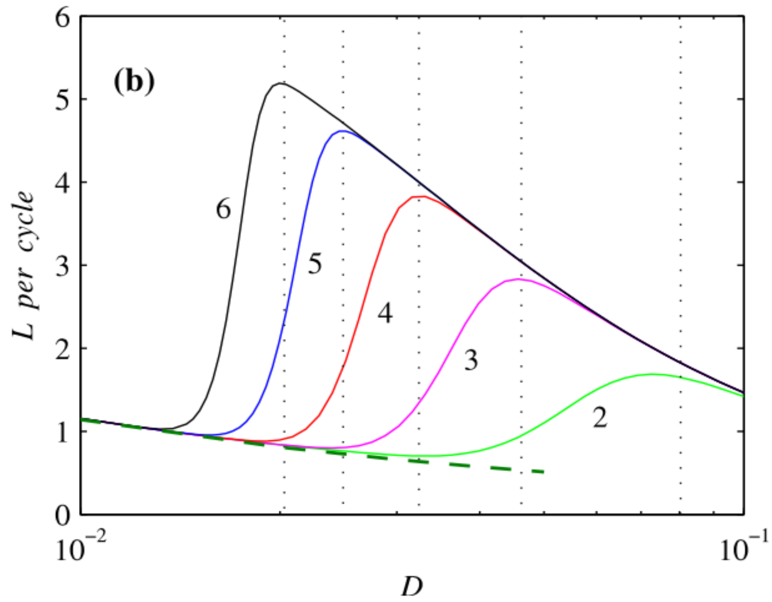
$$E(t) = \frac{dL}{dt}$$



$\mathcal{E}(t)$ as a function of time throughout the period T. All three panels are for $\omega = 10^{-4}$, (a) $D = 0.01$, (b) $D = 0.0324$, (c) $D = 0.1$, and $A = 0.02$, 0.04 and 0.08 as indicated by the numbers beside individual curves.

Information length *L(t) - continued*





$\mathcal{L}$ over one cycle, as functions of the noise level D. The numbers 2 to 6 beside individual curves correspond to $\omega = 10^{-2}$ to 10^{-6}. (a) $A = 0.02$, (b) $A = 0.04$. The thick dashed curves are $\mathcal{L} = 4A/1.4D^{1/2}$. As in Fig. 3, the dotted vertical lines are at D_{res} given by (9) for $\omega = 10^{-2}$ to 10^{-6}; note how well these values again agree with the maxima of the corresponding curves.

4. Conclusions

- Investigated a stochastic resonance by direct numerical solutions of the Fokker-Planck equation.
- Stochastic resonance was shown to be accompanied by a rapid change in PDF, generating a new source of information, which is beautifully captured by the information length.
- Information length is a useful diagnostic tool, with a very clear signature of the resonance emerging.
- In contrast, relative entropy, Kullback-Leibler divergence, etc would not be useful for these problems as they compare only the two probability density functions (e.g. at the two different times).

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