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Entropy Production and Efficiency in Longitudinal Convecting-Radiating Fins †

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Abstract: The properties of the entropy production in convecting-radiating fins are analyzed. By taking advantage of the explicit expression for the distribution of heat along the fin, we investigate the possibility to assess the efficiency of these devices through the amount of entropy produced in the heat transfer process. The analysis is performed both for purely convecting fins and for convecting-radiating fins. A comparison with standard definitions of efficiency is given.

Keywords: entropy production; efficiency; longitudinal fin; convection; radiation; non-linear ODEs

1. Introduction

The longitudinal fins are widely used in applications to enhance heat dissipation from a given device or from a suitable surface. The main mechanisms of heat dissipation are conduction, convection and radiation. While for the first two mechanisms, making the appropriate simplification assumptions (e.g., thermal coefficients independent of temperature), the mathematical models of temperature distribution along the fin are linear, if radiation is added, the models become intrinsically nonlinear and their analysis very challenging. In this work we investigate the role of entropy in assessing the efficiency of the fin. We introduce a novel indicator of the ability of a fin to dissipate heat taking into account the rate of entropy produced by the fin in its steady state. The entropy rate considered here results from the contribution of convection and radiation. The evaluation of the efficiency of a fin with an arbitrary general profile take advantage of the explicit analytical results for the distribution of the temperature in convective-radiative fins obtained in [1], some of which are reported here for ease of reading.

The work is organized as follows: in Section 2 we introduce the main equations describing the evolution of the temperature along the fin and the corresponding boundary conditions. The rate of entropies produced by convection and radiation by the fin are also introduced. In Section 3 an entropy-based indicator for the effectiveness of the fin to dissipate heat is introduced and discussed. For definiteness, the application of the method to some relevant cases is illustrated. In Section 4 the formulae introduced are applied to the case of purely convective fin: the efficiency of a rectangular fin is calculated and a comparison with the classical results from literature is given. In Section 5 the case of a fin dissipating by convection and radiation is presented. Finally, in the conclusions, we discuss our results and their possible generalizations.

2. The Entropy Production Due to Heat Exchange

We consider a longitudinal fin of arbitrary profile attached to a base at a temperature T_b. The fin length is L, whereas the fin thickness at a distance x from the base is $2f_0(x) \geq 0$. The half thickness at the base is $f_b = f_0(x = 0)$, whereas at the fin tip, located at $x = \ell$, the half thickness is denoted by

$f_t = f_0(\ell)$ (see Figure 1). We assume that the Fourier law of heat conduction holds inside the fin and that the temperature varies only along the x direction. The variation of the internal energy is assumed to be equal to the energy gains (or losses) by conduction, radiation and convection.

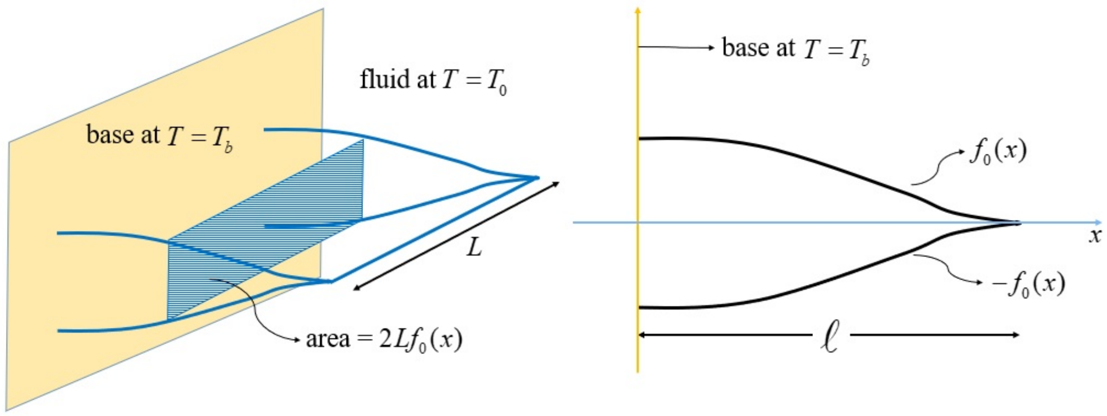

Figure 1. The longitudinal fin with a profile described by a suitable $f_0(x)$ with the coordinate system, the cross-sectional area and the geometrical properties. The case shown corresponds to $f_t = f_0(\ell) = 0$.

If ρ is the density of the homogeneous material, c its specific heat, κ the thermal conductivity, h the convective heat transfer coefficient and σ the Stefan-Boltzmann constant, the evolution of temperature $T(x,t)$ is governed by the following equation:

$$\rho c f_0(x)\frac{\partial T}{\partial t} = \kappa\frac{\partial}{\partial x}\left(f_0(x)\frac{\partial T}{\partial x}\right) - 2h\left(1+2\frac{f_0}{L}\right)(T-T_0) - 2\sigma\epsilon\left(1+2\frac{f_0}{L}\right)(T^4-T_1^4), \tag{1}$$

where T_0 is the temperature of the fluid adjacent to the fin and T_1 represents the temperature of the effective radiation environment (i.e., the radiant energy absorbed by the fin per unit of time and surface is $\epsilon\sigma T_1^4$) and ϵ is the emissivity of the fin.

If the fin thickness is small compared to its length, then the term f_0/L can be ignored and we obtain

$$\rho c f_0(x)\frac{\partial T}{\partial t} = \kappa\frac{\partial}{\partial x}\left(f_0(x)\frac{\partial T}{\partial x}\right) - 2h(T-T_0) - 2\sigma\epsilon(T^4-T_1^4) \tag{2}$$

In the following we assume the fin to be in general non-gray, with $T_1^4 = kT_0^4$, where k is the ratio between the absorptivity and the emissivity of the fin [2]. For a gray fin one has to set $k = 1$ [2].

Equation (2) must be supplied with the initial and boundary conditions: we assume the boundary conditions to be given by [1]

$$f_0(x)\left.\frac{dT}{dx}\right|_{x=0} - \eta_0(T-T_b)|_{x=0} - \xi_0\left.(T^4-kT_b^4)\right|_{x=0} = 0. \tag{3}$$

and

$$f_0(x)\left.\frac{dT}{dx}\right|_{x=\ell} + \eta_1(T-T_0)|_{x=\ell} + \xi_1\left.(T^4-kT_0^4)\right|_{x=\ell} = 0. \tag{4}$$

where η_i and ξ_i, $i = 0,1$, are positive constants proportional to the Biot and radiation-conduction numbers of the ends of the fin. The initial condition is given by $T|_{t=0} = T(x,0) = T_{in}(x)$.

We are interested in the entropy production due to heat exchange, so we assume that the main contribution to the entropy production comes from convection and radiation. The entropy produced by the friction of the fluid on the fin has been considered elsewhere (see e.g., [3]). For a process starting from a temperature distribution at $t = 0$ given by $T_{in}(x)$ up to the temperature $T(x,t)$ at some time $t > 0$, the contribution at x to the entropy production due to the convection is given by $2h(L+2f_0)\ln(T/T_{in})$. Hence, for the entire fin we obtain

$$\dot{s}_h|_{T_{in}\to T} = \int_0^{\ell} 2h(L+2f_0)\ln\left(\frac{T}{T_{in}}\right)dx \tag{5}$$

The contribution to the entropy production due to the radiation can be explicitly calculated under suitable assumptions. For completeness, we report the main formulae in the next lines; for more details the reader can see for example [5]-[8] and [11]-[13]. We assume that the surface of the fin is diffuse gray [2], i.e., it absorbs a fixed fraction of incident radiation for any direction and at any frequency and emits a fixed fraction of the blackbody radiation. For a blackbody radiation, the mean occupation number for the photon gas is given by

$$< n >= \frac{1}{e^{\frac{h\nu}{K_B T}}-1} \tag{6}$$

where h is the Planck constant and K_B the Boltzmann constant. The density of states per unit volume and per unit solid angle is given by

$$\rho(\nu) = \frac{g\nu^2}{c^3} \tag{7}$$

where c is the speed of light and g is the degeneracy factor which takes into account the two possible polarizations of the photons: it is equal to 2 for unpolarized photons (like in our case) and equal to 1 for polarized photons. The contribution to the entropy for each given frequency ν can be written as

$$s(v) = K_B\left((1+<n>)\ln(1+<n>) - <n>\ln(<n>)\right). \tag{8}$$

Equation (8), together with (6) and (7) give, for the total entropy of the blackbody radiation

$$S = \frac{8\pi V K_B^4 T^3}{h^3c^3}\int_0^{\infty} x^2\left[\left(1+\frac{1}{e^x-1}\right)\ln\left(1+\frac{1}{e^x-1}\right) - \left(\frac{1}{e^x-1}\right)\ln\left(\frac{1}{e^x-1}\right)\right]dx \tag{9}$$

In this case the integral can be evaluated explicitly: indeed, with an integration by parts, we get

$$\int_0^{\infty} x^2\left[\left(1+\frac{1}{e^x-1}\right)\ln\left(1+\frac{1}{e^x-1}\right) - \left(\frac{1}{e^x-1}\right)\ln\left(\frac{1}{e^x-1}\right)\right]dx = \frac{1}{3}\int_0^{\infty} x^4\frac{e^x}{(e^x-1)^2}dx. \tag{10}$$

The integral on the right can be evaluated thanks to the following identity (see e.g., [4])

$$I_\alpha(y) \doteq \int_0^{\infty}\frac{x^\alpha}{e^{x-y}-1}dx = \alpha!\sum_{k=1}\frac{e^{ky}}{k^{\alpha+1}}. \tag{11}$$

For simplicity we can assume $\alpha \in \mathbb{R}^+$ and $y \in \mathbb{R}^-$. By taking the derivative of $I_4(y)$ with respect to y and evaluating it to 0 we get

$$\int_0^{\infty} x^4\frac{e^x}{(e^x-1)^2}dx = 4!\sum_{k=1}^{\infty}\frac{1}{k^4} = \frac{4}{15}\pi^4 \tag{12}$$

giving

$$S = \frac{32\pi V K_B^4 T^3\pi^5}{45h^3c^3} = \frac{16\sigma}{3c}VT^3. \tag{13}$$

This result, limited to the blackbody radiation when the number of photons is in equilibrium, is well-known (see e.g., [5]). Due to the interaction of the radiation with matter the number of photons is no longer conserved and the mean occupation number is reduced, for example, by the processes of absorption, emission and reflection (see e.g., [6]). As a consequence, the spectral energy irradiance is reduced too. This reduction is accounted for by the emissivity ϵ of the material, so we can write

$$< n_\epsilon >= \frac{\epsilon}{e^{\frac{h\nu}{K_B T}} - 1}. \tag{14}$$

By repeating all the steps linking Equation (6) for $< n >$ to Equation (13) for S for the balckbody radiation, we get, in the case of a diffuse gray material with emissivity ϵ

$$S_\epsilon = \frac{16\sigma}{3c} I(\epsilon) V T^3 \tag{15}$$

where $I(\epsilon)$ is a dimensionless integral giving the dependence of the radiation entropy by emissivity, explicitly given by

$$I(\epsilon) = \int_0^\infty x^2 \left[\left(1 + \frac{\epsilon}{e^x - 1}\right) \ln\left(1 + \frac{\epsilon}{e^x - 1}\right) - \left(\frac{\epsilon}{e^x - 1}\right) \ln\left(\frac{\epsilon}{e^x - 1}\right) \right] dx. \tag{16}$$

The entropy rate for unit surface $d\dot{s}$ is obtained from (15) as [7,8]

$$d\dot{s} = \frac{16\sigma}{3} I(\epsilon) T^3 \tag{17}$$

From (5) and (17) it follows that the total contribution to the entropy production (in W/°K) of the fin by convection and radiation can be written as

$$\dot{s}|_{T_{in}\to T} = \dot{s}_h|_{T_{in}\to T} + \dot{s}_\sigma|_{T_{in}\to T} = \int_0^\ell 2(L + 2f_0) \left(h \ln\left(\frac{T}{T_{in}}\right) + \frac{16\sigma}{3} I(\epsilon)(T^3 - T_{in}^3) \right) dx \tag{18}$$

and, using the same approximation as in Equation (2), it follows

$$\dot{s}|_{T_{in}\to T} = 2L \int_0^\ell \left(h \ln\left(\frac{T}{T_{in}}\right) + \frac{16\sigma}{3} I(\epsilon)(T^3 - T_{in}^3) \right) dx. \tag{19}$$

For further convenience it is appropriate to introduce dimensionless variables. In particular, let $z = x/\ell$ and $\tau = \kappa t/\rho c \ell^2$ denote the dimensionless coordinates. Moreover we define $\theta = T/T_b$, $\theta_{in} = T_{in}/T_b$, $\alpha = 2h\ell^2/(f_b\kappa)$, $\beta = 2\sigma\epsilon\ell^2 T_b^3/(f_b\kappa)$ and $f(z) = f_0(z\ell)/f_b$. Equation (2) becomes

$$f(z)\frac{\partial\theta}{\partial\tau} = \frac{\partial}{\partial z}\left(f(z)\frac{\partial\theta}{\partial z} \right) - \alpha(\theta - \theta_0) - \beta(\theta^4 - k\theta_0^4) \tag{20}$$

with initial conditions $\theta(z,\tau)|_{\tau=0} = \theta(z,0) = \theta_{in}(z)$ and boundary conditions

$$\begin{aligned} f(z)\frac{d\theta}{dz}\Big|_{z=0} - Bi_0(\theta - 1)|_{z=0} - N_0(\theta^4 - k)\Big|_{z=0} = 0, \\ f(z)\frac{d\theta}{dz}\Big|_{z=1} + Bi_1(\theta - \theta_0)|_{z=1} + N_1(\theta^4 - k\theta_0^4)\Big|_{z=1} = 0, \end{aligned} \tag{21}$$

where the Biot numbers $Bi_j = \eta_j\ell/f_b$, $j = 0, 1$, and the radiation-conduction numbers $N_j = \xi_j\ell/f_b$, $j = 0, 1$ were introduced.

3. The Role of Entropy in Assessing Efficiency of the Steady-State

A common indicator of the capability of a fin to dissipate heat is given by the efficiency [2,9,10]. To define this efficiency, it was necessary to introduce a reference state given by the fin at constant temperature equal to the base temperature T_b ($\theta = 1$). Accordingly, the efficiency η of the fin is defined as the ratio of the actual heat transfer to the ideal heat transfer for a fin of infinite thermal conductivity in the reference state. It can be shown that, for the steady state solution of Equation (20), the efficiency can be calculated as [1]

$$\eta = \frac{Bi_1(\theta_0 - \theta(1)) + Bi_0(1-\theta(0)) + N_1(k\theta_0^4 - \theta(1)^4) + N_0(k - \theta(0)^4)}{\alpha(1-\theta_0) + \beta(1-k\theta_0^4)}. \tag{22}$$

In order to make a comparison with the efficiency as above defined, the calculation of the entropy production $\dot{s}$ is performed by taking the same reference state. Hence, $\dot{s}$ is given by

$$\dot{s} := \dot{s}|_{T\to T_b} = \dot{s}|_{T_{in}\to T_b} - \dot{s}|_{T_{in}\to T} \tag{23}$$

and applying (19) it follows

$$\dot{s} = 2L\int_0^{\ell}\left(\frac{16\sigma}{3}I(\epsilon)(T_b^3 - T^3) - h\ln\left(\frac{T}{T_b}\right)\right)dx. \tag{24}$$

In order to get clearer formulae, we introduce the reference entropy production due to convection, $\dot{s}_{0,h}$, and the reference entropy production due to radiation, $\dot{s}_{0,\sigma}$, as follows:

$$\dot{s}_{0,h} = 2L\ell h, \qquad \dot{s}_{0,\sigma} = 2L\ell\frac{16\sigma}{3}I(\epsilon)T_b^3, \tag{25}$$

so that the expression of the total entropy production reduces to

$$\dot{s} = \int_0^1 \left(\dot{s}_{0,\sigma}(1-\theta^3) - \dot{s}_{0,h}\ln(\theta)\right)dz. \tag{26}$$

Notice that the entropy rate $\dot{s}_{0,\sigma}$ corresponds to the entropy produced by a fin at $\theta = 0$ (i.e., $T = 0$), whereas the entropy rate $\dot{s}_{0,h}$ corresponds to the entropy produced by a fin at $\theta = \exp(-1) \sim 0.368$ (i.e., $T \sim 0.368T_b$).

As pointed out in [1], a large class of steady state solutions of Equation (20) with the boundary conditions (21) are such that the dimensionless temperature $\theta(z)$ is bounded from below by the (dimensionless) fluid temperature, $\theta_0 = T_0/T_b$, and from above by the (dimensionless) base temperature, $\theta_b = T_b/T_b = 1$, i.e., $\theta \in (\theta_0, 1)$.

We are now able to define an entropy-based indicator for the effectiveness of the fin to dissipate heat by convection and radiation. This can be done by defining

$$\eta_s = 1 - \frac{\int_0^1 \left(\dot{s}_{0,\sigma}(1-\theta^3) - \dot{s}_{0,h}\ln(\theta)\right)dz}{\left(\dot{s}_{0,\sigma}(1-\theta_0^3) - \dot{s}_{0,h}\ln(\theta_0)\right)}. \tag{27}$$

If $\theta(z) = \theta_0$, then $\eta_s = 0$, whereas $\eta_s = 1$ when $\theta(z) = \theta_b = 1$. We notice that the ratio of the reference entropies $\dot{s}_{0,\sigma}$ and $\dot{s}_{0,h}$ is related to the ratio of the dimensionless convective and radiative coefficients α and β (see the definitions before Equation (20)) through the formula

$$\frac{\dot{s}_{0,\sigma}}{\dot{s}_{0,h}} = \frac{16}{3}\frac{I(\epsilon)}{\epsilon}\frac{\beta}{\alpha} \tag{28}$$

so that Equation (27) can be written also in the following form

$$\eta_s = 1 - \frac{\int_0^1 \left(\frac{16}{3}\frac{I(\epsilon)}{\epsilon}\beta(1-\theta^3) - \alpha\ln(\theta)\right)dz}{\left(\frac{16}{3}\frac{I(\epsilon)}{\epsilon}\beta(1-\theta_0^3) - \alpha\ln(\theta_0)\right)} \tag{29}$$

In the next section, we will investigate the reliability of this definition by analyzing the purely convective case and making a comparison with the classical definition of efficiency (22).

4. Analysis of the Pure Convective Case

In this section we take into account a fin dissipating heat solely through the convective mechanism. In this case Formula (27) reduces to

$$\eta_s = 1 - \frac{1}{\ln(\theta_0)} \int_0^1 \ln(\theta) dz \tag{30}$$

The simplest case is that of a rectangular longitudinal profile, meaning $f(z) = 1$ for the dimensionless profile. The steady state temperature $\theta(z)$, solution of the Equation (20) with the boundary conditions (21), has been given in [1] as

$$\theta(z) = \theta_0 + (1-\theta_0) Bi_0 \frac{m \cosh(m(1-z)) + Bi_1 \sinh(m(1-z))}{m(Bi_0 + Bi_1)\cosh(m) + (m^2 + Bi_0 Bi_1)\sinh(m)}. \tag{31}$$

From the previous formula it is possible to get the temperature distribution along a fin with an insulated tip and a base at $T = T_b$. When N_0 and N_1 are both zero, the boundary condition corresponding to an insulated tip is obtained from (21) by taking $Bi_1 = 0$, whereas the boundary condition corresponding to a base at $T = T_b$ is obtained by taking the limit $Bi_0 \to \infty$. If this is the case, Equation (31) reduces to

$$\theta(z) = \theta_0 + (1-\theta_0) \frac{\cosh(m(1-z))}{\cosh(m)}. \tag{32}$$

By Equation (30), the corresponding value of entropic efficiency is

$$\eta_s = -\frac{1}{\ln(\theta_0)} \int_0^1 \ln\left(1 + a\cosh(my)\right) dy, \tag{33}$$

where $a = \frac{1-\theta_0}{\cosh(m)\theta_0}$. It is interesting to look at what happens when θ_0 is close to 1. In the limit $\theta_0 \to 1$, it is possible to show that

$$\eta_s = \frac{\tanh(m)}{m} + \frac{1}{8} \frac{\sinh(2m) - 2m}{2m(1+\cosh(2m))}(1-\theta_0) + O((1-\theta_0)^2) \tag{34}$$

For a temperature profile given by (32), the classical efficiency (22) (i.e., the ratio of the actual heat transfer to the ideal heat transfer for a fin of infinite thermal conductivity) is then given by [1,9]

$$\eta = \frac{\tanh(m)}{m} \tag{35}$$

so that Formula (34) can be rewritten as

$$\eta_s = \eta + \frac{1}{8} \frac{\sinh(2m) - 2m}{2m(1+\cosh(2m))}(1-\theta_0) + O((1-\theta_0)^2) \tag{36}$$

From this example it is apparent that (30) can be seen as an extension of the classical definition of the efficiency based on the quantity of heat dissipated by the fin. In Figure 2 we plot the Formula (33) as a function of θ_0 and m. For comparison, the Gardner's result (35) is also reported.

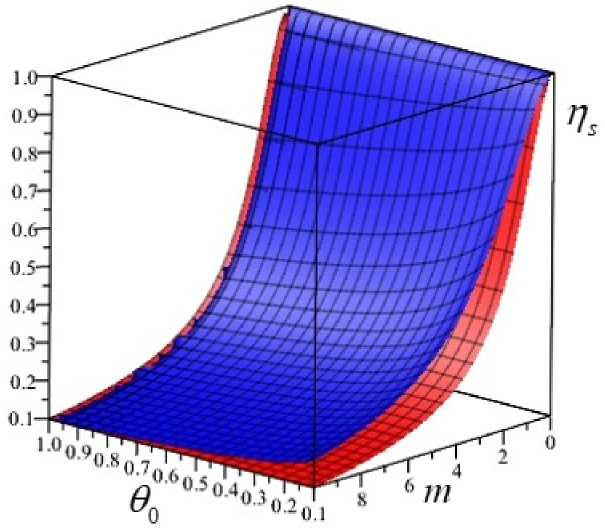

Figure 2. The plot of the efficiency η_s (33) as a function of θ_0 and m (blue) and the classical Gardner's Formula (35) (in red).

A further support to the above point of view is given by looking at the efficiency corresponding to the more general profile temperature (31) in the same limit $\theta_0 \to 1$. Now, from formulae (31) and (33) we get

$$\eta_s = Bi_0 \frac{Bi_1(\cosh(m) - 1) + m \sinh(m)}{m\,(m(Bi_0 + Bi_1)\cosh(m) + (m^2 + Bi_0 Bi_1)\sinh(m))} + O(1 - \theta_0). \tag{37}$$

Again, the first term of this expansion is exactly the efficiency obtained by applying the classical definition of efficiency (22) (see [1], where the classical expressions of efficiency for different other profiles have been given).

In the next section we will look at the more general convective-radiative case.

5. Entropic Efficiency in the Convecting-Radiating Fin

The case of a fin dissipating both by convection and radiation is more challenging since the differential equation, describing the steady state temperature along the fin, is nonlinear and the general solution of the differential equation cannot be written explicitly. In [1] the authors have been able, thanks to a change of variables, to write down (in terms of an auxiliary function $y(z)$) a family of explicit solutions to Equation (20) in the steady case with the boundary conditions (21). For the sake of completeness we report the main formulae and restrict the discussion to gray fins (i.e., we set $k = 1$ in Equations (20) and (21)).

If the change of variables

$$\theta(z) = \theta_0 + wy(z)^2, \quad w \in \mathbb{R}, \tag{38}$$

is inserted into the steady version of Equation (20), and if the further constrain $f(z)\frac{dy}{dz} = 1$ is assumed, then the resulting differential equation can be integrated to give the following implicit formula for $y(z)$ [1]

$$-\frac{A}{y} + E_1 \arctan\left(\frac{y}{b_1}\right) + E_2\left(\arctan\left(\frac{y + a_2}{b_2}\right) + \arctan\left(\frac{y - a_2}{b_2}\right)\right) + F_2 \ln\left(\frac{(y + a_2)^2 + b_2^2}{(y - a_2)^2 + b_2^2}\right) = \frac{\beta w^3}{2}(z + c), \tag{39}$$

where the values of A, E_1, E_2 and F_2 are given by:

$$E_1 = -\frac{1}{b_1^3\left(a_2^2+(b_1-b_2)^2\right)\left(a_2^2+(b_1+b_2)^2\right)},$$
$$E_2 = \frac{1}{4}\frac{\left((a_2^2-3b_2^2)(a_2^2+b_1^2-b_2^2)-2b_2^2(3a_2^2-b_2^2)\right)}{\left(a_2^2+(b_1-b_2)^2\right)\left(a_2^2+(b_1+b_2)^2\right)\left(a_2^2+b_2^2\right)^3 b_2},$$
$$A = \frac{1}{b_1^2(a_2^2+b_2^2)^2},\quad F_2 = \frac{1}{8}\frac{\left((b_2^2-3a_2^2)(a_2^2+b_1^2-b_2^2)+2a_2^2(3b_2^2-a_2^2)\right)}{\left(a_2^2+(b_1-b_2)^2\right)\left(a_2^2+(b_1+b_2)^2\right)\left(a_2^2+b_2^2\right)^3 a_2}.$$

In these expressions the coefficients b_1, a_2 and b_2 are explicit functions of the dimensionless fluid temperature θ_0, the ratio α/β and the parameter w appearing in (38). More explicitly one has:

$$b_1 = \sqrt{\frac{2\theta_0+b}{w}},\quad a_2^2 = \frac{b-2\theta_0+2\sqrt{2\theta_0^2+b^2}}{4w},\quad b_2^2 = \frac{2\theta_0-b+2\sqrt{2\theta_0^2+b^2}}{4w}. \tag{40}$$

where the value of b is fixed by the unique real solution of the following cubic equation,

$$\frac{\alpha}{\beta} = b(b^2+2b\theta_0+2\theta_0^2). \tag{41}$$

At this point it remains to fix the values of the constant c, appearing in (39), and w, appearing in (38). They can be fixed by exploiting the boundary condition at $z=0$. Indeed, it is possible to show (see [1]) that the first of the two boundary conditions (21) can be written as the following polynomial equation for $y(0) = y|_{z=0}$,

$$Bi_0(wy(0)^2-(1-\theta_0)) + N_0\left((wy(0)^2+\theta_0)^4-1\right) - 2wy(0) = 0. \tag{42}$$

Further, for fixed values of the parameters Bi_0, N_0, θ_0 and w, this equation possesses always one real negative solution (see [1]), say y_-. The initial condition for y is then $y(0) = y_-$.

For consistency with the assumed constraints $f(z)\frac{dy}{dz} = 1$ and $f(0) = 1$, it is possible to show that the value of w must be fixed by the following equation

$$f(0) = 1 = \frac{2}{\beta y_-^2\left(wy_-^2+2\theta_0+b\right)\left(\left(wy_-^2+\theta_0-\frac{b}{2}\right)^2+\theta_0(\theta_0+b)+\frac{3}{4}b^2\right)}, \tag{43}$$

whereas the value of c is fixed by Equation (39) evaluated at $y=y_-$ and $z=0$. Consequently, the values of y as a function of z are implicitly determined by Equation (39) for any choice of the parameters β, a and b (i.e., of the parameters θ_0, α and β of the steady version of the differential Equation (20)). Through Equation (38), these functions give the corresponding values of the dimensionless temperature in the steady state $\theta(z)$.

Now we will apply the methodology reported above to describe the dependence of the entropic efficiency (29) on the dimensionless convection and radiation coefficients α and β and on the emissivity ϵ.

For simplicity we analyze the case of a fin with a base at $T=T_b$, i.e., $\theta(0)=1$, corresponding to $Bi_0 \to \infty$ and/or $N_0 \to \infty$. In this case the value of w can also be written as

$$w = \frac{1}{2}\left(\alpha(1-\theta_0)+\beta(1-\theta_0^4)\right). \tag{44}$$

To fix the ideas we can assume the emissivity ϵ to be equal to 0.5. The corresponding value of the integral $I(\epsilon)$ (16) is given by $I(\epsilon) \sim 5.097$. We first choose two different values of θ_0: $\theta_0 = 0.1$ and

$\theta_0 = 0.5$. For each of these choices we consider four different values of α, namely $\alpha = \{0.1, 0.5, 1, 2\}$, and twenty different values of β, from $\beta = 0.1$ to $\beta = 2$. Then, according to Equations (38) and (39) we calculate the distribution of temperatures along the fin, corresponding to a given set of parameters. Finally, we obtain the amount of entropic efficiency of each state by means of Equation (29). The results are reported in Figures 3 and 4: in all cases the efficiency decreases with increasing β and decreases with increasing α. The resulting behavior of η_s is in agreement with that of the classical efficiency (22) by performing similar variations of the parameters. For comparison we report in Figure 5 the values of the efficiency calculated with Formula (22) (given in [1]) by using the same choices of the parameters as above.

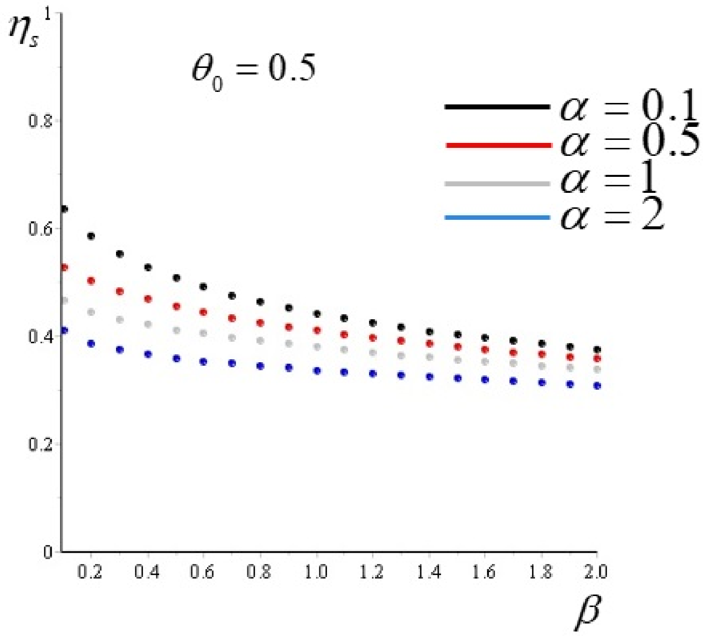

Figure 3. The plot of the efficiency η_s as a function of β for $\theta_0 = 0.5$ and four different values of α.

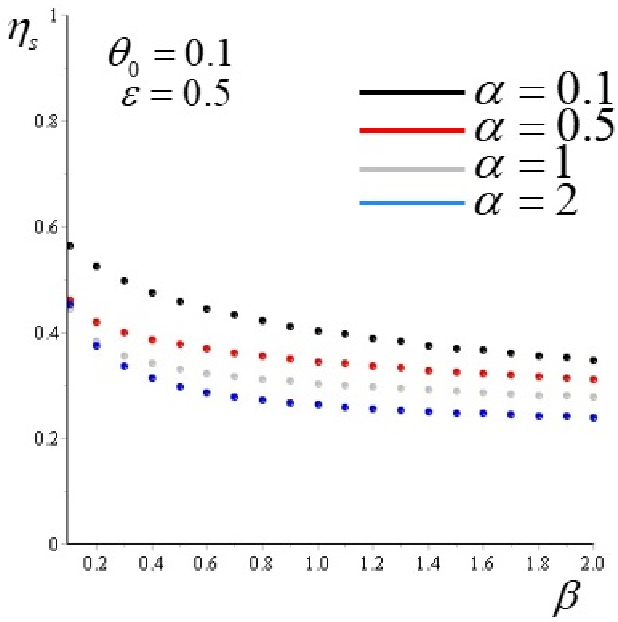

Figure 4. The plot of the efficiency η_s as a function of β for $\theta_0 = 0.1$ and four different values of α.

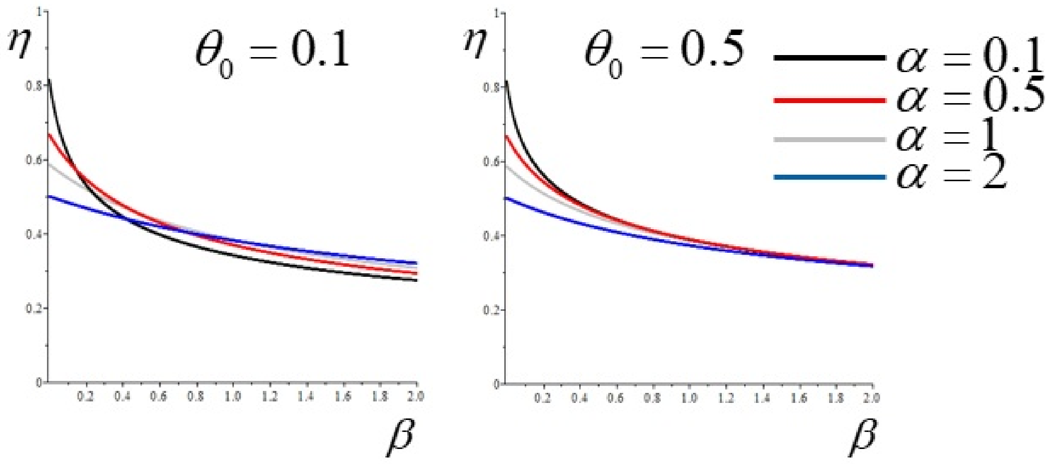

Figure 5. The plot of the classical efficiency η (from [1]) as a function of β for $\theta_0 = 0.1$ and $\theta_0 = 0.5$ and four different values of α.

6. Conclusions

We introduced a novel indicator giving the efficiency of the performances of longitudinal fins of arbitrary profile based on the amount of entropy produced by the fin in its steady state. The contributions to the entropy taken into account have been those coming from convection and radiation. It has been shown that this definition gives values of efficiency that are compatible, in a first approximation, to those given by the classical definition of efficiency based on the analysis of the heat transfer by convection and radiation. In our opinion our definition is however more flexible: the role of the fluid temperature is explicit and this is particularly evident for example from Equation (36). In order to perform the analysis of both the convective and the full convective-radiative cases, we took advantage of the results appeared in [1], where explicit steady solutions of the relevant equations for the distribution of temperature along the fin have been obtained. This work can be considered a starting point for a more in-depth analysis of the efficiency of fins with different profiles and with different mechanisms of heat dissipation. The methodology developed here is fairly general and, although it has been applied to a few simple cases here, it is worth taking into consideration and applied to more complex cases.

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