Entropy production and efficiency in longitudinal convecting-radiating fins

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Introduction and motivations

In this work we investigate the role of the entropy in assessing the capability of longitudinal fins, with an arbitrary profile, to dissipate heat.

The heat is assumed to be transferred by two mechanisms: by thermal convection and by thermal radiation.

Due to the presence of radiation, the mathematical models of temperature distribution along the fin are non-linear and the analysis is challenging.

We take advantage of the explicit analytical results for the distribution of the temperature in convective-radiative fins obtained elsewhere (see [F. Zullo et al., Appl. Math. Model., 2020]).

The mathematical model

We consider a longitudinal fin of arbitrary profile attached to a base at a temperature T_b. The fin length is L, whereas the fin thickness at a distance x from the base is $2f_0(x) \geq 0$. The half thickness at the base is $f_b = f_0(x = 0)$, whereas at the fin tip, located at $x = \ell$, the half thickness is denoted by $f_t = f_0(\ell)$. We assume that the Fourier law of heat conduction holds inside the fin and that the temperature varies only along the x direction. The variation of the internal energy is assumed to be equal to the energy gains (or losses) by conduction, radiation and convection.

The mathematical model

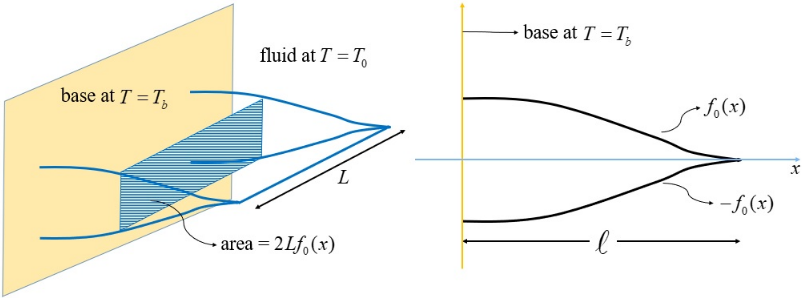

Figure: The longitudinal fin with a profile described by a suitable $f_0(x)$ with the coordinate system, the cross-sectional area and the geometrical properties. The case shown corresponds to $f_t = f_0(\ell) = 0$.

The mathematical model

If ρ is the density of the homogeneous material, c its specific heat, κ the thermal conductivity, h the convective heat transfer coefficient, σ the Stefan-Boltzmann constant and ϵ the emissivity of the fin, the evolution of temperature $T(x,t)$ is governed by the following equation:

$$\rho c f_0(x)\frac{\partial T}{\partial t} = \kappa\frac{\partial}{\partial x}\left(f_0(x)\frac{\partial T}{\partial x}\right) - 2h(T - T_0) - 2\sigma\epsilon(T^4 - T_1^4) \quad (1)$$

We are interested in the entropy production due to heat exchange, so we assume that the main contribution to the entropy production comes from convection and radiation. The entropy produced by the friction of the fluid will be neglected here; for such contribution see e.g. [Xie et al., J. of Heat Transf., 2015].

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The role of the entropy

For a process starting from a temperature distribution at $t = 0$ given by $T_{in}(x)$ up to the temperature $T(x, t)$ at some time $t > 0$, the contribution to the entropy production (in W/K) due to the convection and radiation can be shown to be

$$\dot{s}|_{T_{in}\to T} = 2L\int_0^{\ell}\left(h\ln\left(\frac{T}{T_{in}}\right) + \frac{16\sigma}{3}I(\epsilon)(T^3 - T_{in}^3)\right)dx. \quad (2)$$

where the first addend on the right is the contribution of the convection and the second addend is the contribution of the radiation.
Here $I(\epsilon)$ is an explicit dimensionless integral giving the dependence of the radiation entropy by emissivity.

The role of the entropy

A common indicator of the capability of a fin to dissipate heat is given by the classical efficiency (see e.g. [Howell et al., Thermal Radiation Heat Transfer, 2016]. To define this efficiency, it is necessary to introduce a reference state given by the fin at constant temperature equal to the base temperature T_b. Then, the efficiency of the fin is defined as the ratio of the actual heat transfer to the ideal heat transfer for a fin of infinite thermal conductivity in the reference state.

In order to make a comparison with the classical efficiency as above defined, we perform the calculation of the entropy production $\dot{s}$ by taking the same reference state:

$$\dot{s} := \dot{s}|_{T \to T_b} = \dot{s}|_{T_{in} \to T_b} - \dot{s}|_{T_{in} \to T} \qquad (3)$$

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The role of the entropy

From the previous definition, we explicitly get

$$\dot{s} = 2L \int_0^{\ell} \left(\frac{16\sigma}{3} I(\epsilon)(T_b^3 - T^3) - h \ln\left(\frac{T}{T_b}\right)\right) dx. \qquad (4)$$

To get clearer formulae, we introduce the reference entropy production due to convection, $\dot{s}_{0,h}$, and the reference entropy production due to radiation, $\dot{s}_{0,\sigma}$, as follows:

$$\dot{s}_{0,h} = 2L\ell h, \qquad \dot{s}_{0,\sigma} = 2L\ell \frac{16\sigma}{3} I(\epsilon) T_b^3, \qquad (5)$$

so that the expression of the total entropy production reduces to

$$\dot{s} = \frac{1}{\ell} \int_0^{\ell} \left(\dot{s}_{0,\sigma} \left(1 - (\frac{T}{T_b})^3\right) - \dot{s}_{0,h} \ln\left(\frac{T}{T_b}\right)\right) dx. \qquad (6)$$

It is also convenient to introduce the dimensionless fin depth $z = x/\ell$ and the dimensionless temperature $\theta = T/T_b$. Also, $\theta_0 = T_0/T_b$ is the dimensionless temperature of the fluid and $\theta_b = T_b/T_b = 1$ that of the base.

We define the entropy-based indicator for the effectiveness of the fin to dissipate heat by convection and radiation as:

$$\eta_s = 1 - \frac{\int_0^1 \left(\dot{s}_{0,\sigma}(1-\theta^3) - \dot{s}_{0,h}\ln(\theta)\right) dz}{\left(\dot{s}_{0,\sigma}(1-\theta_0^3) - \dot{s}_{0,h}\ln(\theta_0)\right)}. \tag{7}$$

Notice that if $\theta(z) = \theta_0$, then $\eta_s = 0$, whereas $\eta_s = 1$ when $\theta(z) = \theta_b = 1$.

The purely convective case

To start with, we consider a fin dissipating heat solely through the convective mechanism. In this case the formula for the efficiency reduces to

$$\eta_s = 1 - \frac{1}{\ln(\theta_0)} \int_0^1 \ln(\theta) dz \tag{8}$$

For a fin with an insulated tip and a base at $T = T_b$, the value of the entropic efficiency can be shown to be

$$\eta_s = -\frac{1}{\ln(\theta_0)} \int_0^1 \ln\left(1 + a\cosh(my)\right) dy, \tag{9}$$

where $a = \frac{1-\theta_0}{\cosh(m)\theta_0}$. For comparison, the classical efficiency is (Gardner's formula)

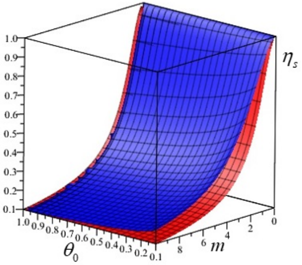
$$\eta = \frac{\tanh(m)}{m} \tag{10}$$

The purely convective case

It is interesting to notice that, for θ_0 close to 1 one has:

$$\eta_s = \frac{\tanh(m)}{m} + \frac{1}{8}\frac{\sinh(2m) - 2m}{2m(1+\cosh(2m))}(1-\theta_0) + O((1-\theta_0)^2) \quad (11)$$

From this formula it is evident that (8) can be seen as an extension of the classical definition of the efficiency based on the quantity of heat dissipated by the fin. In the next figure we plot the formula (9) as a function of θ_0 and m (in blue). For comparison, the Gardner's result (in red) is also reported.



The convective-radiative case

The case of a fin dissipating both by convection and radiation is more complex. The dimensionless differential equation giving the steady state dimensionless temperature $\theta(z)$ is

$$\frac{d}{dz}\left(f(z)\frac{d\theta}{dz}\right) = \alpha(\theta - \theta_0) + \beta(\theta^4 - k\theta_0^4) \tag{12}$$

where α and β are dimensionless convective and radiative coefficients, given by $\alpha = 2h\ell^2/(f_b\kappa)$ and $\beta = 2\sigma\epsilon\ell^2 T_b^3/(f_b\kappa)$.

In [F. Zullo et al., Appl. Math. Model., 2020] the authors give explicit solutions to the previous equation with suitable general boundary conditions (see the paper accompanying these slides for more details): we will use those results to calculate the entropic efficiency and to make a comparison with the classical efficiency.

The convective-radiative case

To fix the ideas we can assume the emissivity ϵ to be equal to 0.5. Also, for the sake of simplicity, we analyze the case of a fin with a base at $T = T_b$, i.e. $\theta(0) = 1$. We first choose two different values of θ_0: $\theta_0 = 0.1$ and $\theta_0 = 0.5$. For each of these choices we consider four different values of α, namely $\alpha = \{0.1, 0.5, 1, 2\}$, and twenty different values of β, from $\beta = 0.1$ to $\beta = 2$. Then, thanks to the explicit formulae of the distribution of temperatures along the fin we obtain the amount of entropic efficiency of each state from (7). The results are reported in the next figures

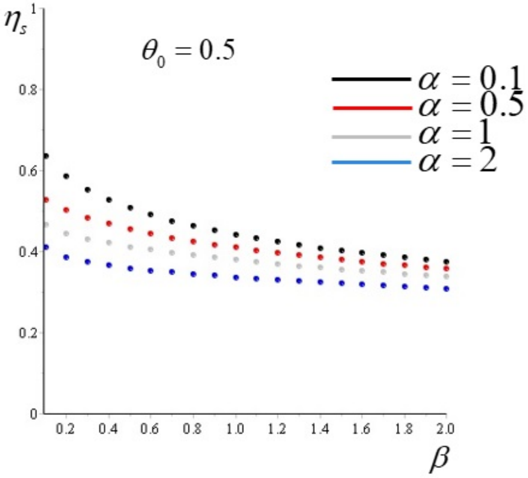


Figure: The plot of the efficiency η_s as a function of β for $\theta_0 = 0.5$ and four different values of α

The convective-radiative case

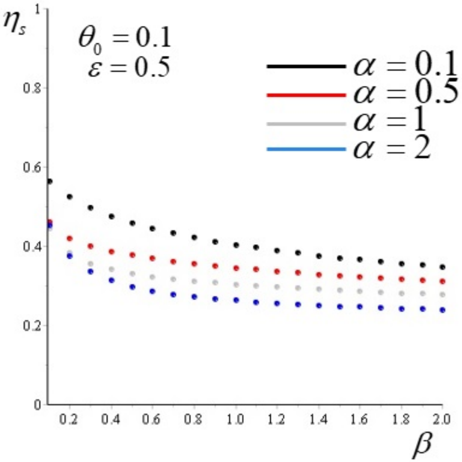


Figure: The plot of the efficiency η_s as a function of β for $\theta_0 = 0.1$ and four different values of α

The convective-radiative case

For comparison we report in the next figure the values of the classical efficiency by using the same choices of the parameters as above.

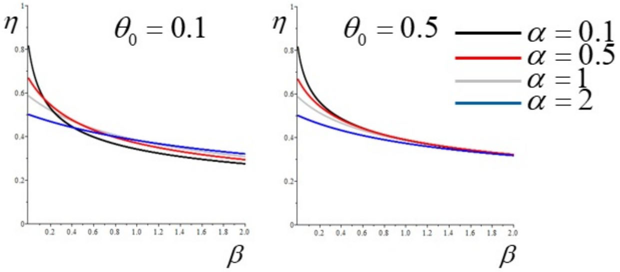


Figure: The plot of the classical efficiency as a function of β for $\theta_0 = 0.1$ and $\theta_0 = 0.5$ and four different values of α.

Conclusions

We introduced a novel indicator giving the efficiency of the performances of longitudinal fins of arbitrary profile based on the amount of entropy produced by the fin in its steady state. The results shows that the definition gives values of efficiency that are compatible, in a first approximation, to those given by the classical definition of efficiency based on the analysis of the heat transfer by convection and radiation. In our opinion our definition is however more flexible because the role of the fluid temperature is explicit (this is particularly evident from the first example). The methodology developed here is fairly general and, although it has been applied to a few cases here, it represents a starting point for a more in-depth analysis of the efficiency of fins with different profiles and with different mechanisms of heat dissipation.

Thanks!