Nonlinear Filter for a System with Randomly Delayed Measurements and Inputs

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Objective

An extension of the conventional CQKF, in order to enable it for dealing with randomly delayed measurements and inputs.

Key points

- Filters estimate the states of a dynamic system recursively from given noisy measurements.
- In networked control systems (NCSs), State estimators and plants are remotely located.
- Measurements and input signals are transmitted through a common unreliable network.
- In such scenario, measurements and input signals are generally delayed.
- Delay is random and of arbitrary step.

Problem Formulation

State space model

- Process model

$$x_k = \phi(x_{k-1}) + B\tilde{u}_k + \eta_{k-1}, \tag{1}$$

- Measurement model

$$z_k = \gamma_k(x_k) + \nu_k, \tag{2}$$

where

- $x_k \in \mathbb{R}^n$ is the state, and $z_k \in \mathbb{R}^d$ is measurement,
- $\phi(x_k)$ and $\gamma(x_k)$ are nonlinear functions of state,
- $\tilde{u}_k$ is delayed input, and B is a matrix with acceptable dimension,
- $\eta_{k-1} \sim \mathcal{N}(0, Q_{k-1})$ and $\nu_k \sim \mathcal{N}(0, R_k)$ are noises.

Modeling of Delayed Measurement

Randomly delayed measurement (y_k)

$$y_k = \sum_{i=0}^{N-1} \beta^{(j,i)} z_{k-i}, \tag{3}$$

where

- $\beta^{(j,i)} = (\prod_{j=0}^{i} \beta_j)(1 - \beta_{i+1})$ and $\beta_0 = 1$,
- β_j ($j = 0, 1, 2, \cdots$) are mutually independent Bernoulli random variables.

- $Y_k = \{y_i\}$ with $\{i = 1, 2, \cdots, k\}$ denotes the set of delayed measurement.
- From the distribution of β_j,

$$\begin{aligned} P(\beta_j = 1) &= p = E[\beta_j], \\ P(\beta_j = 0) &= 1 - p, \\ \text{and} \quad E[(\beta_j - p)^2] &= p(1 - p). \end{aligned}$$

Randomly delayed control input ($\tilde{u}_k$)

$$\tilde{u}_k = \sum_{i=0}^{N-1} \alpha^{(j,i)} u_{k-i}, \tag{4}$$

where

- $\alpha^{(j,i)} = (\prod_{j=0}^{i} \alpha_j)(1 - \alpha_{i+1})$ and $\alpha_0 = 1$,
- $\alpha_j, j \in \{0, 1, 2, \cdots\}$ are mutually independent Bernoulli random variables.

- From the distribution of α_j,

$$\begin{aligned} P(\alpha_j = 1) &= q = E[\alpha_j], \\ P(\alpha_j = 0) &= 1 - q, \\ \text{and} \quad E[(\alpha_j - q)^2] &= q(1 - q). \end{aligned}$$

Assumption

- States follow a first order Markov process

$$P(x_k|x_{k-1}, x_{k-2}, \cdots, x_0) = P(x_k|x_{k-1})$$

- $P(x_k|Y_k)$ is Gaussian with mean $\hat{x}_{k|k}$, and Covariance $P_{k|k}$.
- The prior density function of x_k

$$P(x_k|Y_{k-1}) = \mathcal{N}(x_k; \hat{x}_{k|k-1}, P_{k|k-1}).$$

- The prior density function for delayed measurement y_k is Gaussian, *i.e.*

$$P(y_k|Y_{k-1}) = \mathcal{N}(y_k; \hat{y}_{k|k-1}, P^{yy}_{k|k-1}).$$

- The density function for the non-delayed measurement, $P(z_k|Y_{k-1})$, is Gaussian with mean $\hat{z}_{k|k-1}$ and covariance $P^{zz}_{k|k-1}$ respectively.

State estimation

Generally, state estimation is realized in two steps: (i) time update (ii) measurement update.

Time update

Prior estimated state ($\hat{x}_{k|k-1}$) and its covariance ($P_{k|k-1}$) are

$$\begin{aligned}\hat{x}_{k|k-1} &= E[\{\phi(x_{k-1}) + B\tilde{u}_k + \eta_{k-1}\}|Y_{k-1}] \\ &= \int \phi(x_{k-1})\mathcal{N}(x_{k-1}; \hat{x}_{k-1|k-1}, P_{k-1|k-1})dx_{k-1} + BE[\tilde{u}_k],\end{aligned} \tag{5}$$

$$\begin{aligned}&P_{k|k-1} \\ &= E[\{(x_k - \hat{x}_{k|k-1})(x_k - \hat{x}_{k|k-1})^T\}|Y_{k-1}] \\ &= E[\phi(x_{k-1})\phi^T(x_{k-1})|Y_{k-1}] - E[\phi(x_{k-1})|Y_{k-1}]E[\phi^T(x_{k-1})|Y_{k-1}] \\ &\quad + BE[\tilde{u}_k\tilde{u}_k^T]B^T - BE[\tilde{u}_k]E[\tilde{u}_k^T]B^T + Q_k.\end{aligned} \tag{6}$$

Measurement update

The expectation and covariance of the non-delayed measurement

$$\hat{z}_{k|k-1} = \int \gamma(x_k)\mathcal{N}(x_k; \hat{x}_{k|k-1}, P_{k|k-1})dx_k, \tag{7}$$

$$P^{zz}_{k|k-1} = \int \gamma(x_k)\gamma^T(x_k)\mathcal{N}(x_k; \hat{x}_{k|k-1}, P_{k|k-1})dx_k - \hat{z}_{k|k-1}\hat{z}^T_{k|k-1} + R_k. \tag{8}$$

The cross-covariance between state and non-delayed measurement

$$P^{xz}_{k|k-1} = \int x_k\gamma^T_k(x_k)\mathcal{N}(x_k; \hat{x}_{k|k-1}, P_{k|k-1})dx_k - \hat{x}_{k|k-1}\hat{z}^T_{k|k-1}. \tag{9}$$

Measurement update

The expectation and covariance of the delayed measurement

$$\hat{y}_{k|k-1} = \sum_{i=0}^{N-1} (\prod_{j=1}^{i} p_j)(1 - p_{i+1})\hat{z}_{k-i|k-i-1}. \tag{10}$$

$$P^{yy}_{k|k-1} = \sum_{i=0}^{N-1} (\prod_{j=1}^{i} p_j)(1 - p_{i+1}) P^{zz}_{k-i|k-i-1} + \sum_{i=0}^{N-1} (\prod_{j=1}^{i} p_j)(1 - p_{i+1}) \\ \{1 - (\prod_{j=1}^{i} p_j)(1 - p_{i+1})\}\hat{z}_{k-i|k-i-1}\hat{z}^T_{k-i|k-i-1}. \tag{11}$$

The cross-covariance between state and delayed measurement

$$P^{xy}_{k|k-1} = \sum_{i=0}^{N-1} (\prod_{j=1}^{i} p_j)(1 - p_{i+1}) P^{xz}_{k-i|k-i-1}. \tag{12}$$

State estimation

The posterior state estimate

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k(y_k - \hat{y}_{k|k-1}), \quad (13)$$

and the posterior error covariance

$$P_{k|k} = P_{k|k-1} - K_k P^{yy}_{k|k-1} K_k^T, \quad (14)$$

where the Kalman gain (K_k) is

$$K_k = P^{xy}_{k|k-1}(P^{yy}_{k|k-1})^{-1}. \quad (15)$$

Algorithm diagram for Bayesian framework of filtering

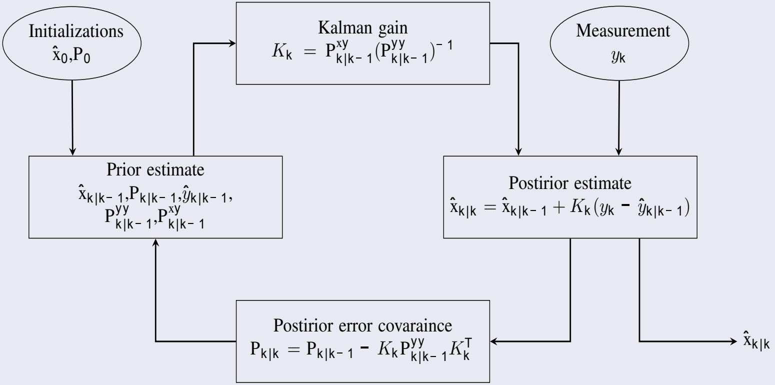


Figure 1: Algorithm diagram for nonlinear filtering under Bayesian framework

Problem 1

Here we consider a dynamic system with state equation:

$$x_k = 2cos(x_{k-1}) + B\tilde{u}_k + \eta_k,$$

and the measurement equation:

$$y_k = \sqrt{1 + x_k^T x_k} + \nu_k,$$

where $\eta_k \sim \mathcal{N}(0,\ Q_k)$ and $\nu_k \sim \mathcal{N}(0,\ R_k)$.

- The following parameters are used for simulation: $u_k = 2sin(0.2k)$,$Q_k = 5I_6$ and $R_k = 5$.
- The filter is initialized with x_0=$0.1_{6\times1}$, $\hat{x}_{0|0}$=$15_{6\times1}$ and $P_{0|0}$=$5I_6$.
- Simulation has been carried out for 200 time-steps.
- Average RMSE is calculated over 200 MC runs for different values of p (we assume $p = q$).

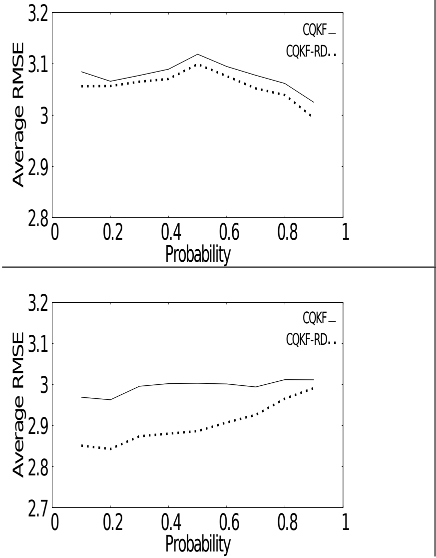
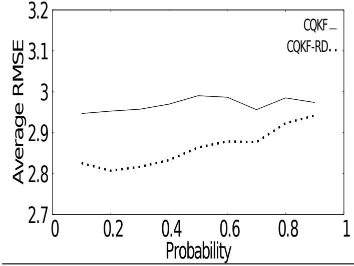
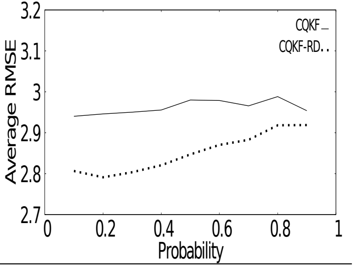


Figure 2: Average RMSE against probability plot, for (a) state-1 (b) state-2 (c) state-3

Simulation Results





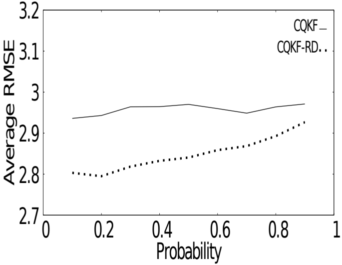


Figure 3: Average RMSE against probability plot, for (a) state-4 (b) state-5 (c) state-6

Discussion and Conclusion

- We have developed a generalized framework of nonlinear filtering in the presence of arbitrary random delay under
  - transmission of measurement from sensor to estimator,
  - transmission of input from controller to system.
- The methodology is realized with the CQKF and the proposed CQKF (CQKF-RD).
- Performance of the filtering method is observed using averaged RMSE.
- The superiority of the proposed method (CQKF-RD) has been shown over CQKF.

Selected References

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Thank you!