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Introducing quantum mechanics in high schools: a proposal based on Heisenberg's *Umdeutung*

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Abstract: Teaching and learning QM at high-school as well as undergraduate level is a highly non trivial task. Indeed major changes are required in understanding the new physical reality, and students have to deal with counterintuitive concepts such as uncertainty and entanglement and advanced mathematical tools. In order to overcome these critical issues a simple approach is here presented, which is based only on 2-vectors and 2x2 matrix algebra. As a further bonus, it could make also possible to fill the gap between high school curricula and the actual scientific and technological advances in physics by allowing students to gain some insight into topics such as qubits and quantum computers. The inspiring source of our proposal, as well as its firm theoretical foundation, can be found in the famous *Umdeutung* (reinterpretation) paper by W. Heisenberg, which introduces QM in matrix form.

Keywords: quantum mechanics; physics teaching; two-level systems; history of physics

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1. Introduction

At high school as well as undergraduate level, Quantum Mechanics (QM) is usually introduced through an overview of the main crucial experiments and theoretical attempts which took place at the beginning of 20-th century. Even if retracing the historical path which led to the introduction of the new conceptual and mathematical framework has undoubted advantages, there are also significant drawbacks, mainly in contexts, such as a high school, where students' lack of advanced mathematical tools puts severe constraints to the understanding of quantum concepts.

On the other hand, QM implies major changes in understanding the world and the physical reality. Introducing concepts such as probability, uncertainty and superposition, and discussing issues such as non locality and entanglement, is a highly non trivial task. Students have to face with a matter, which is counterintuitive and in conflict with the usual classical view of the physical world [1]. In this respect, high school students' difficulties in accepting nondeterminism have been recently recognized [2] to induce a fall back to classical reasoning and a subsequent misunderstanding of the concept of quantum states. Last but not least, the introduction of wave functions and Schroedinger equation, even for the simplest paradigmatic examples of infinite square well and harmonic oscillator, implies the solution of second order ordinary differential equations, which are usually beyond high school standard students' knowledge in calculus.

All the above considerations lead us to think that a better strategy could be to concentrate the attention to two-level systems, which live in a finite dimensional Hilbert space. That allows us to introduce, from the very beginning, a simple 2x2 matrix formulation of QM, where quantum states are identified with 2-vectors belonging to a finite vector space and observables are 2x2 matrices. In this way students have the possibility to become familiar with the unique conceptual issues of QM, such as superposition principle, non locality and entanglement without an advanced mathematical background.

That could make also possible to fill the gap between high school curricula and the actual scientific and technological advances in physics by allowing students to have a glimpse to modern research topics. In fact two-level systems, under the name of qubits, are the basic building blocks of quantum information and computation. Furthermore, our proposal could be envisaged as a useful supplement to game based and simulation teaching strategies [3,4].

According to the above considerations, two-level systems look like ideal candidates to introduce QM to advanced high-school students. But the other side of the coin is that two-level systems are more tightly linked with the notion of an operator acting on a state and of eigenvalue equations. Thus teachers have to justify the formalism, in particular they should be able to explain why observables are operators (matrices) acting on states and, finally, why measured quantities are identified with eigenvalues of such matrices. Our idea is to look for a motivation in history of physics. In the specific case we find a firm basis for our proposal in Heisenberg's seminal paper of 1925 [5], the so called *Umdeutung* paper, which gives the first recognition of the role of matrices in quantum physics. The aim of Heisenberg paper is to set up quantum theory by building only on observables quantities. In fact he provided calculational rules for computing transition frequencies between stationary states, without reference to unobservable characteristics of such states. These rules were later identified by Born and Jordan [6,7] as matrix operations, while matrices being representations of operators in the basis of eigenvectors of the Hamiltonian [8].

Advantages of Heisenberg's approach in undergraduate teaching have been already put forward by some authors [9-11], but the possibility of using it for advanced high school teaching hasn't been explored yet. Our work aims at filling this gap. The stage for studying two-level systems may be set by following Heisenberg's line of reasoning. Mandatory prerequisite is a basic historical introduction to quantum physics, which includes standard topics such as Planck's hypothesis, the photoelectric effect and Bohr's model of hydrogen atom [12]. A detailed analysis of Heisenberg's paper [5] is beyond the scope of this work but one of its inspiring points, which we assume as a starting point of our proposal, is Bohr's postulate that the frequencies of emitted radiation were proportional to energy differences between two stationary states, not to the orbital frequencies of the electrons (as within classical physics). As a consequence in QM we deal with physical quantities, which depend on two states rather than one. This leads naturally to the introduction of matrices.

The net result of our study is a novel teaching-learning sequence on QM properly designed for advanced high-school students and very useful also for in-service and pre-service teacher training.

2. Introducing operators (matrices): basic steps

In this Section we set the theoretical basis of our proposal by recalling Bohr's postulate and then describing Heisenberg's key ideas in a form suitable to high-school students. We stress main logical steps and discuss a simple example: an harmonic oscillator. By further simplifying, we are led to a toy model with only two levels, a ground state and an excited one, described in terms of 2-vectors and 2x2 matrices.

2.1. Bohr's atomic model

As well known, one of Bohr's postulates [12] includes the hypothesis that an atom can be in one of a series of stationary states, each of which corresponds to a discrete value of energy. While an electron is in one of these states, its energy does not vary, while it can radiate by going from a given state to another one with lower energy, according to the fundamental relation:

$$\nu_{mn} = \frac{E_m - E_n}{h}, \qquad E_m > E_n. \tag{1}$$

Thus frequencies are guaranteed to obey the Rydberg-Ritz combination principle:

$$\nu_{mn} = \nu_{mp} + \nu_{pn}\,, \tag{2}$$

which is experimentally observed. This notation associates to each frequency two indices, one for the starting state of the electron and one for the arriving state.

2.2. Heisenberg's original argument

Two main ideas led Heisenberg to matrix mechanics [5]. First, the recognition that at the atomic scale classical mechanics is not valid any more. Second, the correspondence principle must be valid, and in fact in Heisenberg's approach, each quantum equation has a classical corresponding formula.

The starting point is the consideration that in the quantum realm only transitions between states are observable, hence physical quantities should be associated with two states, rather than one, and thus have two indices. Consider any dynamical quantity $x(t)$, which could be for example the position of a particle; consider then its Fourier representation (for simplicity we limit ourselves to the case in which x is periodic; in the general case all the sums appearing in the following formulas are to be replaced with integrals):

$$x_n(t) = \sum_{j=-\infty}^{+\infty} a_j e^{ij\omega t}\,. \tag{3}$$

If x referred to a single quantum state, for instance a Bohr orbit, the Fourier coefficients a_j would depend on the corresponding quantum number, i.e. $a_j = a_j(n)$, with $\omega = \omega(n) = 2\pi\nu(n)$ being the corresponding angular frequency. However, single quantum states are not observable, rather only transition processes associated with two states are. Hence, Heisenberg replaces $a_j(n) = a(n, n-j)$, and $\omega(n) = \omega(n, n-j)$, with $\omega(n, n-j) = 2\pi\nu_{n,n-j}$, and the resulting expression is:

$$x(t) = \sum_{j=-\infty}^{+\infty} a(n, n-j) e^{ij\omega(n,n-j)t}\,. \tag{4}$$

This is the first "reinterpretation of kinematical relations" by Heisenberg and is suggested by the correspondence principle, according to which quantities related with quantum jumps between two states coincide with quantities related to single states in the limit of large quantum numbers. The next step is to represent in this way products of dynamical quantities, which is required for instance to write down energies. Hence, by generalizing the convolution theorem for Fourier series and transforms, Heisenberg argues that the most natural assumption is to represent the square of $x(t)$ as

$$x^2(t) = \sum_{k=-\infty}^{+\infty} b(n, n-k) e^{ik\omega(n,n-k)t}\,, \tag{5}$$

where, according to the combination principle, $\omega(n, n-k) = \omega(n, n-j) + \omega(n-j, n-k)$, and

$$b(n, n-k) = \sum_{j=-\infty}^{+\infty} a(n, n-j) a(n-j, n-k)\,. \tag{6}$$

This is Heisenberg's rule for multiplying transition amplitudes.

In his paper, Heisenberg proceeds by showing how to find transition amplitudes and frequencies from the dynamics of the system. In particular, he links the coefficients $a(n, n-j)$ to Kramers' dispersion formula [13-15] (which constituted his main inspiration, together with Born's generalization to general systems [16]), which describes the interaction of an atom with electromagnetic radiation, and to Planck's constant h through a reformulation of Sommerfeld's quantization condition. In this way he got the relation:

$$h = 4\pi m \sum_{j=0}^{\infty} \{|a(n+j,n)|^2 \omega(n+j,n) - |a(n,n-j)|^2 \omega(n,n-j)\}\,, \tag{7}$$

which is nothing but the Thomas-Reiche-Kuhn sum rule [17, 18]. He then applies his formalism to a simple system, the anharmonic oscillator, where he can determine the amplitudes and the frequencies by finding and solving some recursion relations which are satisfied by them. The quantities$a(n,n-j)$, and $\omega(n,n-j)$, were recognized by Born [6] to be elements of (infinite-dimensional) matrices, since Eq. (6) is nothing but the row by column product of a matrix with elements a_{nm} with itself. Thus, physical quantities in the Heisenberg scheme as reformulated in [6] correspond to infinite matrices. Moreover, in the same paper [6] Born and Jordan recognize that the allowed energies for a quantum system are given by the diagonal elements of the matrix representing the Hamiltonian, namely, its eigenvalues.

2.3. A simple example: harmonic oscillator

Here the simple case of harmonic oscillator [19] is briefly discussed in order to show Heisenberg's scheme at work in a concrete example, quite easy to be digested by advanced high-school students.

In general the problem can be rephrased in the following way: given a conservative force $F(x)$ that binds the electron in an atom, find the quantum mechanical properties, frequencies ω_{nm} and amplitudes a_{nm}, associated with the transitions between stationary states. For a simple harmonic oscillator the force is $F(x) = -kx$, so that the solution to the equation of motion $F(x) = m\ddot{x}$ is:

$$x(t) = a cos \omega_0 t\,, \tag{8}$$

where a is the fundamental amplitude and $\omega_0 = \sqrt{k/m}$ is the frequency. From Sommerfeld's quantization condition one easily gets $ma^2\omega_0\pi = nh$, which gives the allowed values of the vibration amplitude:

$$a(n) = \sqrt{\frac{2\hbar n}{m\omega_0}}\,, \tag{9}$$

while frequency is independent on n, i.e. $\omega(n) = \omega_0$. Let's now substitute amplitudes $a(n)$ into the classical energy function $E = \frac{1}{2} m\omega_0^2 a^2$ and obtain the quantum energy spectrum:

$$E_n = n\hbar\omega_0\,. \tag{10}$$

There exists a single Fourier term (see Eq. (8)), so that only transitions between adjacent states, $n \to n-1$, are allowed. Finally, correspondence principle allows one to compute the radiation frequency and the transition amplitudes from the expressions for $a(n)$ and $\omega(n)$ above obtained, getting:

$$\omega_{n,n-1} = \omega_0, \qquad a_{n,n-1} = \sqrt{\frac{2\hbar n}{m\omega_0}}\,. \tag{11}$$

These quantities, as Born and Jordan pointed out [6], are identified with elements of matrices.

The simple procedure here shown gives a strong motivation for identifying operators acting on a Hilbert space of quantum states with matrices. At this stage one is naturally led to introduce, for the sake of simplicity, as a toy model a system built of only two levels, i. e. a ground state and an excited state, whose observables are described by 2x2 matrices acting on two-component vectors.

3. Results: playing with two-level systems

In the previous Section we set the stage to outline the core of our teaching-learning sequence. The basic pillar of our proposal is the quantum two-state system, which may be introduced by making explicit reference to concrete physical examples (e.g. a single spin and a measurement apparatus or the polarization of a photon). Then, by taking the single spin system, an identification has to be made between the corresponding space of states and a two-dimensional vector space. This allows one to choose the two basis vectors $|u\rangle$ and $|d\rangle$ as two-component column vectors and to construct a general state as the vector which is a linear superposition of $|u\rangle$ and $|d\rangle$. The single quantum spin is an example of a large class of simple systems called qubits. Indeed a qubit is the basic building block of quantum information and computation, in much the same way as the bit, that is, a binary variable that constitutes the smallest piece of information, is the fundamental brick of classical information theory and current computer science.

The subsequent step deals wth the introduction of physical observables, which are the object of measurement and are conveniently identified as 2x2 matrices within the same vector space. Within the single spin system, the matrix form of spin components can be simply derived and identified with Pauli matrices. Then average values of observables can be easily computed as well as eigenvalues and eigenvectors.

The representation of quantum states and observables as vectors and matrices of a two-dimensional vector space gives the simple machinery upon which peculiar quantum mechanical features can be built up, such as mixed and entangled states in composite systems.

It is now an easy task to illustrate uncertainty principle by making reference to two different components of the spin, as it applies to many pairs of measurable quantities and not only to position and momentum. In this way the deep meaning of a quantum measurement process and its differences with the classical case can be pointed out. The next step is to show how to combine single spins to get composite systems. This amounts to introduce non locality issues, quantum correlations and entanglement, allowing one to gain some insight into unique quantum features.

4. Discussion

Our teaching-learning sequence has been implemented within various training activities in QM held in a bunch of high schools in southern Italy, and aimed to teachers in physics and mathematics as well as to selected students attending the last year of a scientific high school. Preliminary results, gathered by interviews as well as surveys filled by participants both before and after activities, lead us to make the following considerations.

First, teachers and students have the possibility to become familiar with quantum issues such as entanglement and non locality without an advanced mathematical background and this is, indeed, an advantage.

Second, it is possible to fill the gap between high-school curricula and the actual scientific and technogical advances in physics (e.g. qubits, quantum computer, quantum teleportation). This could lead to an increasing number of students, which may choose to undertake scientific university programs.

Third, the proposal is also suitable within pre-service and in-service training programs for physics teachers. Indeed teachers appear much more interested in learning basic principles and practical teaching strategies then in deepening their knowledge of formalism.

Finally, a promising strategy in physics education could be to shape teaching-learning proposals by relying on the historical path, which led to a concept, as well as on the social and philosophical contexts in which the concept itself developed. In this way significant changes in teachers' and students' conceptions regarding the Nature of Science are expected.

5. Conclusions

A novel strategy is here presented, which allows to introduce peculiar features of QM at high school level without resorting to advanced mathematical tools. This non trivial task has been accomplished building upon vectors and matrices in a two-dimensional vector space. The inspiring source of our proposal, as well as its firm theoretical foundation, can be recognized as the 1925 seminal paper by W. Heisenberg. which provides a simple calculational method to deal with quantum mechanical states and observables, based on the identification of the physical quantities of interest with transition frequencies and amplitudes. Indeed such frequencies and amplitudes form matrices. Preliminary results gathered among both high school teachers and students are encouraging and offer useful insights for further improvements. **Supplementary Materials:** The following are available online at www.mdpi.com/xxx/s1, Figure S1: title, Table S1: title, Video S1: title.

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Data Availability Statement: In this section, please provide details regarding where data supporting reported results can be found, including links to publicly archived datasets analyzed or generated during the study. Please refer to suggested Data Availability Statements in section "MDPI Research Data Policies" at https://www.mdpi.com/ethics. You might choose to exclude this statement if the study did not report any data.

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Abbreviations

The following abbreviations are used in this manuscript:
QM: Quantum Mechanics

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