Gravitation in the space with chimney topology

Maxim Eingorn[‡], Andrew McLaughlin II[‡],

Ezgi Canay[†], Maksym Brilenkov[*], Alexander Zhuk[§]

[‡]Department of Mathematics and Physics, North Carolina Central University, Durham, NC, USA

[†]Department of Physics, Istanbul Technical University, Istanbul, Turkey

[*]Institute of Theoretical Astrophysics, University of Oslo, Oslo, Norway

[§] Astronomical Observatory, Odessa I.I. Mechnikov National University, Odessa, Ukraine

ECU2021

Outline

- Introduction
- The gravitational potential
  - the Helmholtz equation
    solution from delta functions
    solution from periodic image contributions
- Numerical point of view
  - accuracy & minimum number of terms
  - comparison of alternative formulas
- Conclusion

Introduction

Spatial topology of the universe →

What is the shape of the space?
Is it positively curved, negatively curved or flat?

Is the universe finite or infinite?

How could the topology have affected the early evolution of the universe in the quantum gravity regime? What was its role in the large scale structure formation at later stages?

General Relativity admits
any type of spatial topology →

Space might be simply connected
(in agreement with concordance cosmology),
or, just as well,
multiply connected.

Introduction

In a multiply connected universe, the volume may be finite even for negative or zero curvature.

P.A.R. Ade et al. [Planck Collaboration], A&A 571 (2014) A26

If the universe covers a much wider region than the observable sector, the finiteness of it cannot be deduced from the current data. For a rather smaller volume, however, it is reasonable to trace observational indications of its shape.

possible topologies include:

$\boldsymbol{T} \times \boldsymbol{R} \times \boldsymbol{R}$ → slab

$\boldsymbol{T} \times \boldsymbol{T} \times \boldsymbol{R}$ → chimney

$\boldsymbol{T} \times \boldsymbol{T} \times \boldsymbol{T}$ → three torus

J.-P. Luminet, arXiv:0802.2236

especially on the CMB

Introduction

CMB anomalies in large angular scale observations may be consequences of the spatial topology.

P. Bielewicz and A.J. Banday, MNRAS 412 (2011) 2104

P. Bielewicz, A.J. Banday and K.M. Gorski, Proceedings of the XLVIIth Rencontres de Moriond, 2012, eds. E. Auge, J. Dumarchez and J. Tran Thanh Van, published by ARISF, p. 91

preferred axis of the quadrupole & octopole alignment
«axis of evil»

→

an equal-sided chimney $\boldsymbol{R} \times \boldsymbol{T}^2$
and
a slab $\boldsymbol{R}^2 \times \boldsymbol{T}$

G. Aslanyan and A.V. Manohar, JCAP 06 (2012) 003
E.G. Floratos and G.K. Leontaris, JCAP 04 (2012) 024

Introduction

From Planck 2013 data,
the radius of the largest sphere
that may be inscribed in the topological domain
is bounded from below by

$R_i > 0.92\chi_{rec}$

$\boldsymbol{T^3}$ (cubic torus)

$R_i > 0.71\chi_{rec}$

$\boldsymbol{R} \times \boldsymbol{T^2}$ (equal-sided chimney)

$R_i > 0.50\chi_{rec}$

$\boldsymbol{R^2} \times \boldsymbol{T}$ (slab)

distance to the recombination surface (of the order of 14 Gpc)

P.A.R. Ade et al. [Planck Collaboration], A&A 571 (2014) A26

Planck 2015 data imposes
the tighter constraints

$R_i > 0.97\chi_{rec}$

$\boldsymbol{T^3}$ (cubic torus)

$R_i > 0.56\chi_{rec}$

$\boldsymbol{R^2} \times \boldsymbol{T}$ (slab)

P.A.R. Ade et al. [Planck Collaboration], A&A 594 (2016) A18

The gravitational potential

Perturbations in discrete cosmology (for the ΛCDM model)

Non-relativistic matter presented as separate point-like particles $\rightarrow \rho = \sum_n m_n \delta(\boldsymbol{r} - \boldsymbol{r}_n)$

- ρ: mass density

Unperturbed FLRW metric:

$$ds^2 = a^2\left(d\eta^2 - \delta_{\alpha\beta} dx^\alpha dx^\beta\right)$$

- η: conformal time
- $a(\eta)$: scale factor
- x^α: comoving coordinates ; $\alpha, \beta = 1,2,3$
- $\kappa \equiv 8\pi G_N / c^4$; G_N: Newtonian gravitational constant
  c: speed of light

Perturbed metric for the inhomogeneous universe:

Weak gravitational field limit } Metric corrections are considered as 1st order quantities.

$$ds^2 = a^2\left[(1 + 2\Phi) d\eta^2 - (1 - 2\Phi)\delta_{\alpha\beta} dx^\alpha dx^\beta\right]$$

- $\Phi(\eta, \boldsymbol{r})$: scalar perturbation (gravitational potential)

The gravitational potential (the Helmholtz equation)

Einstein equations yield:

$$\Delta\Phi - \frac{a^2}{\lambda_{\rm eff}^2}\Phi = \frac{\kappa c^2}{2a}\delta\rho$$

M. Eingorn, ApJ 825 (2016) 84
E. Canay and M. Eingorn, Phys. Dark Univ. 29 (2020) 100565

$$\Phi = \frac{\kappa c^2 \bar{\rho}}{2a^3}\lambda_{\rm eff}^2 + \hat{\Phi} \longrightarrow \Delta\hat{\Phi} - \frac{a^2}{\lambda_{\rm eff}^2}\hat{\Phi} = \frac{\kappa c^2}{2a}\rho$$

$\boldsymbol{\lambda}_{\rm eff}$: the effective screening length specifying the cutoff distance of the gravitational interaction in the cosmological setting

Today $\boldsymbol{\lambda}_{\rm eff}$ is approximately 2.6 Gpc.

- $\Delta \equiv \delta^{\alpha\beta}\partial^2/(\partial x^\alpha \partial x^\beta)$
- $\bar{\rho}$: average mass density ($\bar{\varepsilon} = \bar{\rho}c^2/a^3$)
- $\delta\rho(\eta, \boldsymbol{r}) \equiv \rho - \bar{\rho}$ (mass density fluctuation)

The gravitational potential (solution from delta functions)

In the space with chimney topology $\boldsymbol{T_1} \times \boldsymbol{T_2} \times \boldsymbol{R}$, and
for a particle $\boldsymbol{m}$ placed at the center of Cartesian coordinates,

$$\delta(x) = \frac{1}{l_1} \sum_{k_1=-\infty}^{+\infty} \cos\left(\frac{2\pi k_1}{l_1} x\right), \quad \delta(y) = \frac{1}{l_2} \sum_{k_2=-\infty}^{+\infty} \cos\left(\frac{2\pi k_2}{l_2} y\right),$$

which intrinsically contain the information of the infinitely many periodic images,
located at points shifted from $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = (\mathbf{0}, \mathbf{0}, \mathbf{0})$
by multiples of $\boldsymbol{l_1}$ and $\boldsymbol{l_2}$ along the corresponding axes:
$(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = (\boldsymbol{k_1 l_1}, \boldsymbol{k_2 l_2}, \mathbf{0}),\ \ \boldsymbol{k_{1,2}} = \mathbf{0}, \pm\mathbf{1}, \pm\mathbf{2}, \ldots$

- l_1, l_2: periods of the tori T_1, T_2 along the x- and y-axes, respectively

The gravitational potential (solution from delta functions)

$$\Delta\hat{\Phi} - \frac{a^2}{\lambda_{\text{eff}}^2}\hat{\Phi} = \frac{\kappa c^2}{2a}\rho$$

$$\tilde{\Phi}_{\cos} \equiv \left(-\frac{\kappa c^2}{8\pi a}\frac{m}{l}\right)^{-1}\hat{\Phi}_{\cos} = \sum_{k_1=-\infty}^{+\infty}\sum_{k_2=-\infty}^{+\infty}\left(k_1^2 + k_2^2 + \frac{1}{4\pi^2\tilde{\lambda}_{\text{eff}}^2}\right)^{-1/2}$$

$$\times \exp\left(-\sqrt{4\pi^2(k_1^2+k_2^2) + \frac{1}{\tilde{\lambda}_{\text{eff}}^2}}\,|\tilde{z}|\right)\cos(2\pi k_1\tilde{x})\cos(2\pi k_2\tilde{y})$$

- $x = \tilde{x}l,\ y = \tilde{y}l,\ z = \tilde{z}l,\ \lambda_{\text{eff}} = \tilde{\lambda}_{\text{eff}}al$
- $l_1 = l_2 = l$

The gravitational potential
(solution from periodic image contributions)

$$\Delta\hat{\Phi} - \frac{a^2}{\lambda_{\text{eff}}^2}\hat{\Phi} = \frac{\kappa c^2}{2a}\rho$$

- $x = \tilde{x}l,\ y = \tilde{y}l,\ z = \tilde{z}l,\ \lambda_{\text{eff}} = \tilde{\lambda}_{\text{eff}}al$
- $l_1 = l_2 = l$

$$\tilde{\Phi}_{\text{exp}} \equiv \left(-\frac{\kappa c^2}{8\pi a}\frac{m}{l}\right)^{-1}\hat{\Phi}_{\text{exp}} = \sum_{k_1=-\infty}^{+\infty}\sum_{k_2=-\infty}^{+\infty}\frac{1}{\sqrt{(\tilde{x}-k_1)^2+(\tilde{y}-k_2)^2+\tilde{z}^2}}$$

$$\times \exp\left(-\frac{\sqrt{(\tilde{x}-k_1)^2+(\tilde{y}-k_2)^2+\tilde{z}^2}}{\tilde{\lambda}_{\text{eff}}}\right)$$

Each term (Yukawa potential) in the series corresponds to the individual contribution of one of the infinitely many periodic images.

Numerical point of view
(accuracy & minimum number of terms)

$\widetilde{\Phi}_{\cos}$ and $\widetilde{\Phi}_{\exp}$ consist of infinite series, so it is necessary to know the minimum number n of terms required to calculate them numerically for any order of accuracy.

$n_{\cos}$ $n_{\exp}$

Here, we demand $$\left|\frac{\text{exact } \widetilde{\Phi} - \text{approximate } \widetilde{\Phi}}{\text{exact } \widetilde{\Phi}}\right| < 0.001,$$

and the formula ($\widetilde{\Phi}_{\cos}$ or $\widetilde{\Phi}_{\exp}$) which admits the smaller n serves as a better tool for numerical analysis.

obtained from the formula for $\widetilde{\Phi}_{\exp}$, for $n \gg n_{\exp}$

Numerical point of view
(accuracy & minimum number of terms)

$\widetilde{\Phi}_{\cos}$ and $\widetilde{\Phi}_{\exp}$ both contain double series, thus
n is ascribed the minimum number of combinations (k_1, k_2) to be included in the sequence,
generated in the increasing order of $\sqrt{k_1^2 + k_2^2}$,
to attain the desired precision

When $n \geq n_{\exp}$, the approximate values of $\widetilde{\Phi}_{\exp}$ agree with the exact ones up to 0.1%.

The minimum numbers of terms needed in the $\widetilde{\Phi}_{\cos}$ expression to get the exact potential values, again, up to 0.1%, correspond to $n_{\cos}$.

Numerical point of view (comparison of alternative formulas)

The rescaled gravitational potential $\tilde{\Phi}$ and numbers $n_{\exp}$ and $n_{\cos}$ of terms in series at eight selected points for $\tilde{\lambda}_{\text{eff}} = 0.01$ and $\tilde{\lambda}_{\text{eff}} = 0.1$.

$\tilde{\lambda}_{\text{eff}} = 0.01$:

	$\tilde{x}$	$\tilde{y}$	$\tilde{z}$	$\tilde{\Phi}$	$n_{\exp}$	$n_{\cos}$*
A_1	0.5	0	0.5	5.524×10^{-31}	2	1007
A_2	0.5	0	0.1	2.810×10^{-22}	2	—
A_3	0.5	0	0	7.715×10^{-22}	2	—
B_1	0.1	0	0.5	1.405×10^{-22}	1	187
B_2	0.1	0	0.1	5.101×10^{-6}	1	2119
B_3	0.1	0	0	4.540×10^{-4}	1	—
C_1	0	0	0.5	3.857×10^{-22}	1	236
C_2	0	0	0.1	4.540×10^{-4}	1	1479

$\tilde{\lambda}_{\text{eff}} = 0.1$:

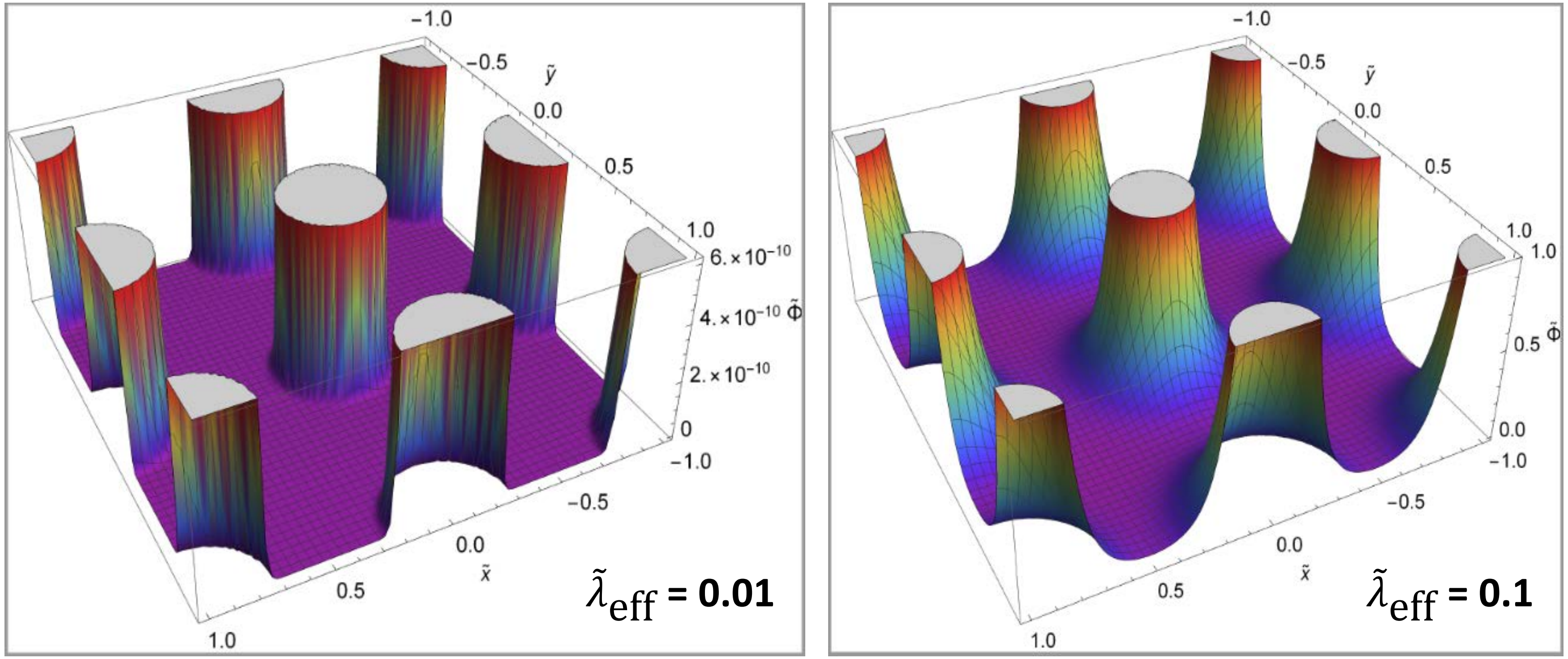
	$\tilde{x}$	$\tilde{y}$	$\tilde{z}$	$\tilde{\Phi}$	$n_{\exp}$	$n_{\cos}$*
A_1	0.5	0	0.5	2.418×10^{-3}	7	40
A_2	0.5	0	0.1	2.398×10^{-2}	6	808
A_3	0.5	0	0	2.700×10^{-2}	4	—
B_1	0.1	0	0.5	1.203×10^{-2}	4	28
B_2	0.1	0	0.1	1.719	1	380
B_3	0.1	0	0	3.679	1	—
C_1	0	0	0.5	1.353×10^{-2}	4	37
C_2	0	0	0.1	3.679	1	490

Wolfram Research, Inc., Mathematica, Version 11.3, Champaign, IL, 2018.

$n_{\exp} \ll n_{\cos}$ → $\tilde{\Phi}_{\exp}$ is a better option than its alternative for reducing the computational cost in numerical analysis

*The dash hides incorrect outputs produced due to computational complications

Rescaled gravitational potential $\widetilde{\Phi}$ for $z = 0$.



Wolfram Research, Inc., Mathematica, Version 11.3, Champaign, IL, 2018.

Conclusion

- For the chimney topology $\boldsymbol{T} \times \boldsymbol{T} \times \boldsymbol{R}$ of the universe, the solution to the Helmholtz equation for the gravitational potential may be presented in two alternative forms:

$$\Delta\hat{\Phi} - \frac{a^2}{\lambda_{\text{eff}}^2}\hat{\Phi} = \frac{\kappa c^2}{2a}\rho$$

  - by Fourier expanding delta functions using periodicity along two toroidal dimensions,
  - as the plain summation of the solutions to the Helmholtz equation, for a source particle and its images, all of which admit Yukawa-type potential expressions.

- The solution containing the series sum of Yukawa potentials is a better choice for use in numerical calculations:
  - the desired accuracy is attained by keeping fewer terms in the series in the physically significant cases, i.e. for $\tilde{\lambda}_{\text{eff}} < 1$.

Thank you!

The work of M. Eingorn and A. McLaughlin II was supported by National Science Foundation (HRD Award #1954454).

Disclaimer: the opinions, findings, and conclusions or recommendations expressed are those of the authors and do not necessarily reflect the views of the National Science Foundation.