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Lambda Perturbations and Instability of Keplerian Orbits in the Expanding Universe

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Introduction

- Since the concept of Dark Energy (*i.e.*, effective Lambda-term in the General Relativity equations) became a commonly-accepted paradigm in cosmology, numerous authors tried to analyze its effects on the dynamics of celestial bodies.
  - Unfortunately, such calculations were usually performed only in the framework of static Schwarzschild−deSitter metric, which does not possess the adequate cosmological asymptotics at infinity. As a result, only the conservative perturbations of the orbits were taken into account.
- The aim of the present work is to use the more realistic Robertson−Walker asymptotics and, thereby, to analyze also the nonconservative (secular) perturbations of the Keplerian orbits.
- As an appropriate mathematical tool, we shall employ the modified Kottler metric, which was derived in our earlier paper [Yu.V. Dumin. *Phys. Rev. Lett.*, v.98, p.059001 (2007)]; and the equations of motion of a test body will be solved in this metric.

Mathematical Formalism - 1

• Schwarzschild solution of the General Relativity equations was generalized to the case of the Lambda-term by Kottler as early as 1918:

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{c^2 r'} - \frac{\Lambda r'^2}{3}\right) c^2 \mathrm{d}t'^2 + \left(1 - \frac{2GM}{c^2 r'} - \frac{\Lambda r'^2}{3}\right)^{-1} \mathrm{d}r'^2 + r'^2 (\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\varphi^2) \; ;$$

and this metric is often called also Schwarzschild–deSitter solution.

• Unfortunately, since the above-written solution was derived well before the birth of the modern cosmology, it suffers from a lack of the adequate cosmological asymptotics at infinity.

− Taking into account such asymptotics should be especially important in the case of Dark-Energy-dominated Universe: Since the Dark Energy (or Lambda-term) is present everywhere, it could affect the motion of celestial bodies, in principle, at any spatial scale.

Mathematical Formalism - 2

- The original Kottler solution was reduced to the Robertson–Walker cosmological coordinates in our earlier work [Yu.V. Dumin, *Phys. Rev. Lett.*, 98, 059001 (2007)]:

$$\mathrm{d}s^2 = g_{tt}\, c^2 \mathrm{d}t^2 + 2\, g_{tr}\, c\, \mathrm{d}t\, \mathrm{d}r + g_{rr}\, \mathrm{d}r^2 + g_{\theta\theta}\, \mathrm{d}\theta^2 + g_{\varphi\varphi}\, \mathrm{d}\varphi^2 \,,$$

$$g_{tt} = \frac{-\left(1 - \frac{r_g}{r'} - \frac{r'^2}{r_0^2}\right)^2 + \left(1 - \frac{r'^2}{r_0^2}\right)^2 \frac{r'^2}{r_0^2}}{\left(1 - \frac{r_g}{r'} - \frac{r'^2}{r_0^2}\right)\left(1 - \frac{r'^2}{r_0^2}\right)^2} \,, \quad g_{rr} = \frac{\left(1 - \frac{r'^2}{r_0^2}\right)^2 - \left(1 - \frac{r_g}{r'} - \frac{r'^2}{r_0^2}\right)^2 \frac{r'^2}{r_0^2}}{\left(1 - \frac{r_g}{r'} - \frac{r'^2}{r_0^2}\right)\left(1 - \frac{r'^2}{r_0^2}\right)^2} \frac{r'^2}{r^2} \,,$$

$$g_{tr} = \frac{\left(1 - \frac{r'^2}{r_0^2}\right)^2 - \left(1 - \frac{r_g}{r'} - \frac{r'^2}{r_0^2}\right)^2}{\left(1 - \frac{r_g}{r'} - \frac{r'^2}{r_0^2}\right)\left(1 - \frac{r'^2}{r_0^2}\right)^2} \frac{r'}{r_0} \frac{r'}{r} \,,$$

$$g_{\theta\theta} = g_{\varphi\varphi} / \sin^2\theta = r'^2,$$

$$r' = a_0 \exp\left(\frac{ct}{r_0}\right) r \,,$$

$$r_g = 2GM/c^2,\ r_0 = \sqrt{3/\Lambda} \,.$$

- Up to the first non-vanishing terms of r_g and $1/r_0$, this metric can be written as

$$g_{tt} \approx -\left[1 - \frac{2GM}{c^2 r}\left(1 - \frac{c\sqrt{\Lambda}\, t}{\sqrt{3}}\right)\right], \quad g_{rr} \approx \left[1 + \frac{2GM}{c^2 r}\left(1 - \frac{c\sqrt{\Lambda}\, t}{\sqrt{3}}\right)\right]\left(1 + \frac{2c\sqrt{\Lambda}\, t}{\sqrt{3}}\right),$$

$$g_{tr} \approx \frac{4\, GM\sqrt{\Lambda}}{\sqrt{3}\, c^2} \,, \quad g_{\theta\theta} = g_{\varphi\varphi}/\sin^2\theta \approx r^2\left(1 + \frac{2c\sqrt{\Lambda}\, t}{\sqrt{3}}\right).$$

Mathematical Formalism - 3

- Equations of motion of a test particle in the field of the massive central body are

$$2\left[1-\frac{r_g}{r}\Big(1-\frac{t}{r_0}\Big)\right]\ddot{t} \;-\; 4\,\frac{r_g}{r_0}\,\ddot{r} \;+\; \frac{r_g}{r_0}\,\frac{1}{r}\,\dot{t}^{\,2} \;+\; 2\,\frac{r_g}{r^2}\Big(1-\frac{t}{r_0}\Big)\,\dot{t}\,\dot{r}$$

$$+\,\frac{1}{r_0}\,\Big(2+\frac{r_g}{r}\Big)\,\dot{r}^{\,2} \;+\; 2\,\frac{r^2}{r_0}\,\Big(\dot{\theta}^{\,2}+\sin^2\!\theta\,\dot{\varphi}^{\,2}\Big) \;=\; 0\,,$$

$$4\,\frac{r_g}{r_0}\,\ddot{t} \;+\; 2\left[1+2\,\frac{t}{r_0}+\frac{r_g}{r}\Big(1+\frac{t}{r_0}\Big)\right]\ddot{r} \;+\; \frac{r_g}{r^2}\Big(1-\frac{t}{r_0}\Big)\,\dot{t}^{\,2} \;+\; \frac{2}{r_0}\Big(2+\frac{r_g}{r}\Big)\,\dot{t}\,\dot{r}$$

$$-\,\frac{r_g}{r^2}\Big(1+\frac{t}{r_0}\Big)\,\dot{r}^{\,2} \;-\; 2\,r\Big(1+2\,\frac{t}{r_0}\Big)\Big(\dot{\theta}^{\,2}+\sin^2\!\theta\,\dot{\varphi}^{\,2}\Big) \;=\; 0\,,$$

$$r\Big(1+2\,\frac{t}{r_0}\Big)\,\ddot{\theta} \;+\; 2\,\frac{r}{r_0}\,\dot{t}\,\dot{\theta} \;+\; 2\,\Big(1+2\,\frac{t}{r_0}\Big)\,\dot{r}\,\dot{\theta}$$

$$-\,r\,\Big(1+2\,\frac{t}{r_0}\Big)\sin\theta\,\cos\theta\,\dot{\varphi}^{\,2} \;=\; 0\,,$$

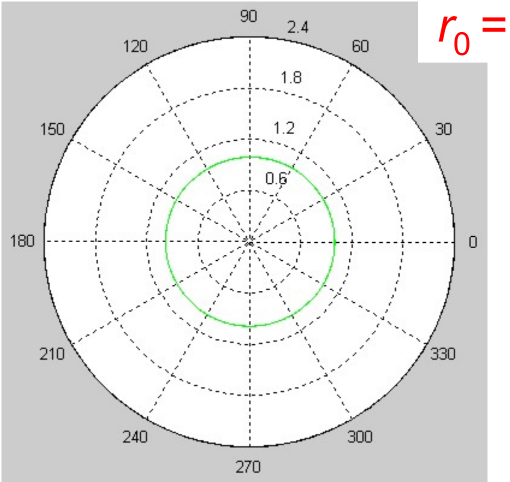
$$r\Big(1+2\,\frac{t}{r_0}\Big)\sin\theta\,\ddot{\varphi} \;+\; 2\,\frac{r}{r_0}\sin\theta\,\dot{t}\dot{\varphi} \;+\; 2\,\Big(1+2\,\frac{t}{r_0}\Big)\sin\theta\,\dot{r}\dot{\varphi}$$

$$+\,2\,r\Big(1+2\,\frac{t}{r_0}\Big)\cos\theta\,\dot{\theta}\dot{\varphi} \;=\; 0\,.$$

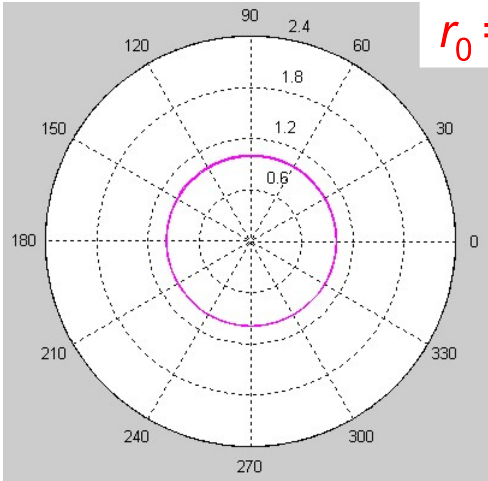
- For example, in the Earth–Moon system: $r_g \sim 10^{-2}$ m, $R_0 \sim 10^{9}$ m, $r_0 \sim 10^{27}$ m; *i.e.,* the characteristic scales differ from each other by many orders of magnitude.

Results of Numerical Integration - 1

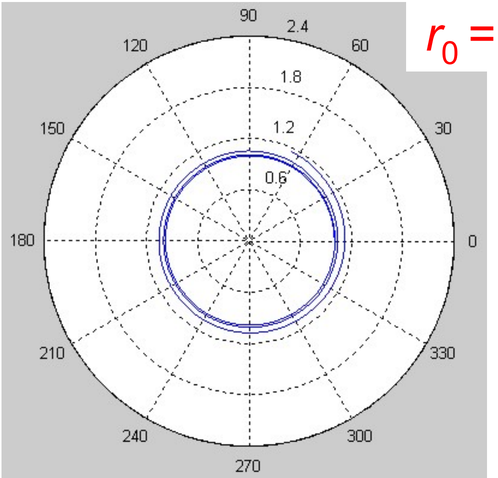
- To simplify calculations, we assume that difference between the characteristic scales of the problem (Schwarzschild radius, the initial orbital radius, and deSitter radius) is not so much as in reality, *e.g.* $\boldsymbol{r_g = 0.01}$, $\boldsymbol{R_0 = 1}$:



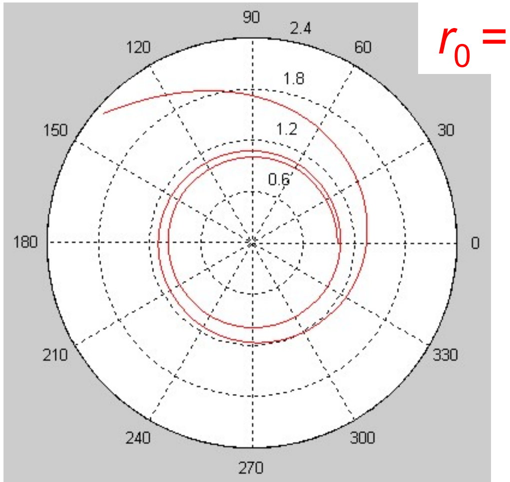
$r_0 = \infty$



$r_0 = 5000$

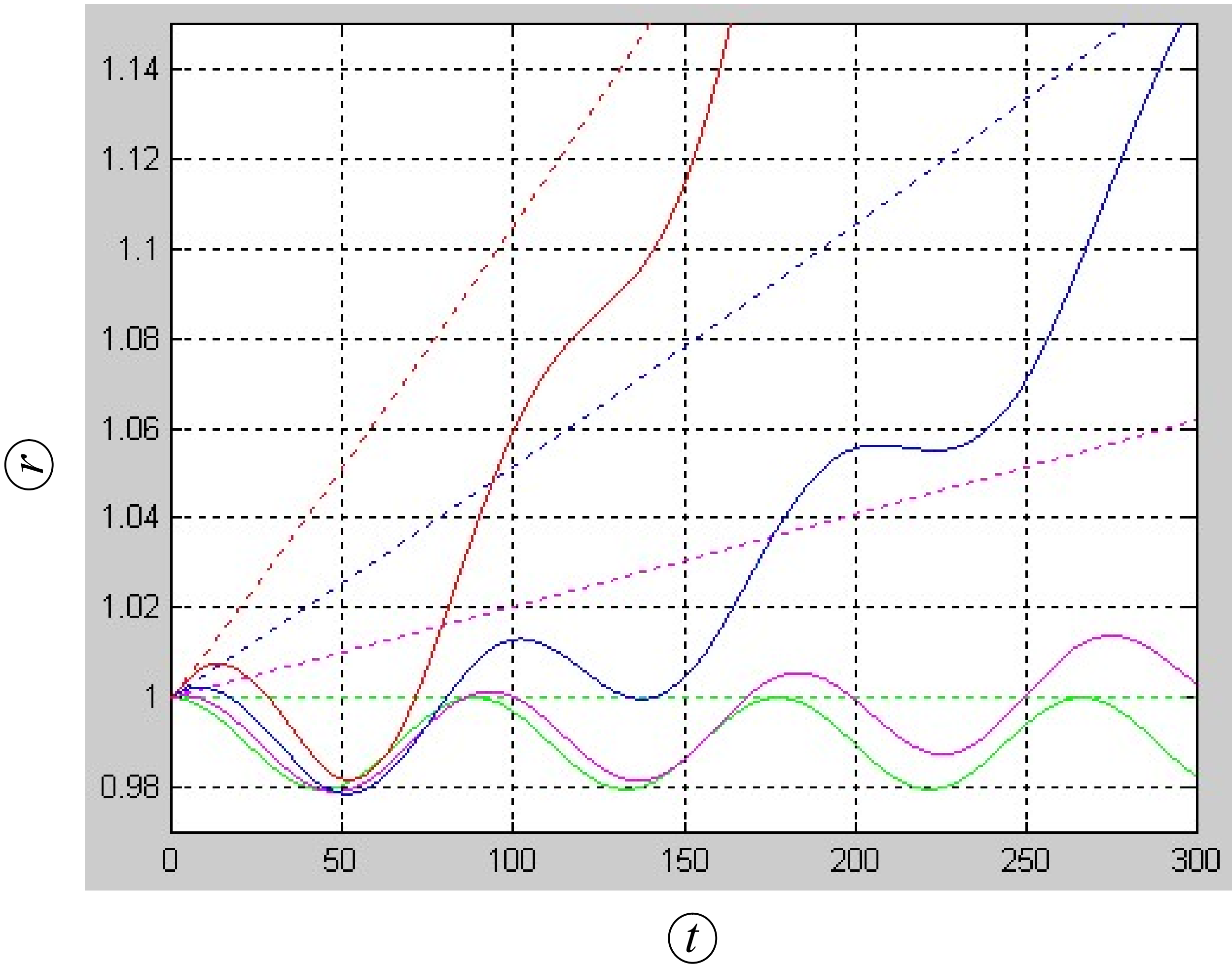


$r_0 = 2000$



$r_0 = 1000$

Results of Numerical Integration - 2



$r_g = 0.01$

$R_0 = 1$

$r_0 = 1000$

$r_0 = 2000$

$r_0 = 5000$

$r_0 = \infty$

Note 1: The curves are wavy because the initial (unperturbed) planetary orbits were taken to be slightly elliptical.

Note 2: Dashed lines represent the standard Hubble flow (unperturbed by the central mass).

- **In certain circumstances, the perturbation caused by the Lambda-term (*i.e.*, Dark Energy) becomes substantial** and even can reach the rate of the standard Hubble flow at infinity**.**

Discussion and Summary

- The problem of cosmological effects at the local (*e.g.,* interplanetary) scales is studied already for almost 90 years but still remains a poorly understood subject (especially, in the case of arbitrary energy–momentum tensor and inhomogeneous background mater distribution).
- A theoretical consideration of the two-body problem becomes much simpler and straightforward in the case of the Lambda-dominated cosmological background.
  - However, the analytical perturbation theory is still unavailable.
  - Besides, the numerical calculations encounter the problem of the very different spatial scales involved.
- Numerical treatment of a few toy models shows that:
  - a perturbation of the test body by the Lambda-term begins to increase quickly at certain values of the parameters (*i.e.*, a kind of the instability develops); and
  - the resulting effect can be significant, namely, the rate of secular increase of the orbital radius can reach the rate of the standard Hubble flow at infinity.
- These facts may have important applications to the long-term dynamics of planetary systems and stellar binaries.