Proceedings

From canonical quantization of lattice gravity through its phase space reduction to anisotropic cosmology

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Version February 22, 2021 submitted to Journal Not Specified

Abstract: For the construction of a generally relativistic quantum field theory, background-independent quantum gravity is needed. By assuming the ADM decomposition of spacetime, it is possible to define the metric-independent discrete analog of a Fock space for quantum gravity on a lattice. This space, known as the spin network, is invariant under the SU(2) symmetry and the spatial diffeomorphisms transformations. It is the states space for loop quantum gravity. The improved construction of the lattice regularization and cosmological reduction of this model is presented in this article. The application of the former procedure to the Hamiltonian constraint provides its lattice analog, the domain of which has a natural structure of a sum of elementary cells. As a result, the related scalar constraint operator, which spectrum is independent of intertwiners, can be defined. The cosmological phase space reduction of lattice gravity requires a rigorous application of gauge-fixing conditions. The obtained Hamiltonian constraint is finite (without any cut-off introduction) and exact (with the holonomy expansion around the unit element of $SU(2)$). It describes a simple structure of inhomogeneities and anisotropies. Consequently, the construction of the quantum evolution of the Universe in terms of transition amplitudes (instead of using perturbative approximations) appears to be possible.

Keywords: canonical quantum gravity; lattice gravity; canonical quantum cosmology; lattice cosmology; gauge representation of gravity; phase space reduction of lattice gravity; loop quantum gravity; loop quantum cosmology

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1. Introduction

Physicists have been trying to construct the hypothetical theory of quantum gravity for decades. One of the recent approaches, formulated in the late nineties of the twentieth century, is canonical loop quantum gravity (LQG) [1–4]. This approach assumes a systems-equivalent (SE), thus metric-independent description in a noncovariant formalism. LQG defines a nonperturbative formalism and aims to achieve background-independent results (by restoring unmodified metric tensor coupling). So far, this last point, however, remains wishful thinking — the (semi)classical limit of this theory has not been calculated. Moreover, contrary to the idea of LQG, several perturbative models based on its formalism have been intensively developing, for instance in [5–7].

The approach proposed in this article postulates to more strictly impose the methodology, which led to the formulation of LQG. In particular, it is required that the SE description will be applicable not only to quantum gravity but to the whole generally relativistic quantum field theory (QFT). Moreover, the quantization is assumed to be constructed in the standard Dirac-Wigner procedure [8,9], known from quantum mechanics. Furthermore, the mathematical formalism that describes real physical processes is restricted to uniformly define all the representatives of the phenomena that belong to the same classes. Finally, all approximations and simplifications are limited as follows. If any quantity is approximated or constrained, all analogous quantities have to be consistently restricted. Moreover, one cannot partially remove this restriction for some objects after some stage of derivations.

This methodological consequence in the construction of the relativistic field theory leads to promising results. Although the generally relativistic QFT of all Standard Model interactions is far from being even sketched, the early outcomes regarding the gravitational field are remarkable — see [10] for a review. By consistently imposing approximations in the construction of gauge fields lattice representations, the structure of results simplifies [11]. In particular, conversely to LQG, all the quantities related to different first-class constraints decouple on a lattice [12]. Moreover, the domain of the Hamiltonian constraint becomes separable [11]. Furthermore, by satisfying the Wigner theorem [9,13], the quantum states space, associated with each link of the lattice, can be defined as the standard Hilbert space for gauge fields [10,14] as in LQG. Finally, by imposing gauge fixing conditions on the lattice, turns out that the resulting cosmological framework describes local anisotropies without the introduction of perturbations [15,16].

The overview of the aforementioned early results of the new program toward the general relativistic formulation of QFT is presented in this article.

2. Methods

The SE formalism of LQG is based on the well-known quantization method of the gauge fields constraint systems [17]. In the case of the Einsteinian gravity [18], the constraints decomposition is revealed by the Legendre transform of the Einstein-Hilbert action of the metric field. Introducing the following decomposition:

$$g_{\mu\nu}dx^\mu dx^\nu = (N^a N_a - N^2)dt^2 + 2N_a dt dx^a + q_{ab}dx^a dx^b \,. \tag{1}$$

one finds that the elements inside the Hamiltonian, which are contracted with N and N_a, are different first-class constraints [19]. To the set of these elements, which generate gauge symmetries known as the temporal and spatial diffeomorphisms, respectively, one can introduce an additional internal invariance. The action proposed in [20], which corresponds to this extension, is formulated in terms of the following gauge fields:

$$A_a^i := \frac{1}{2}\epsilon^{ijk}\Gamma_{jka} + \gamma {\Gamma^i}_{0a} = -2\,\mathrm{tr}\big(A_a \tau_i\big)\,, \qquad E_i^a := \frac{\delta S^{\text{HOLST}}}{\delta \partial_t A_a^i} = \sqrt{q}\, e_i^a = -2\,\mathrm{tr}\big(E^a \tau_i\big)\,, \tag{2}$$

known as the Ashtekar-Barbero variables [21,22]. Here, $\Gamma^\alpha_{\beta a}$ is the Lorentz connection coefficient in the spatial basis, where $\alpha, \beta, ...$ are the Minkowski variables, and $i, j, ...$ describe the related positive Euclidean sector. The decomposition in (1) led to the splitting of the spacetime indices μ, ν into the temporal gauge sector and the spatial dynamical sector. The latter is represented by the metric q_{ab} (q is its determinant). The symbol γ is the real Barbero-Immirzi parameter and the $\mathfrak{su}(2)$ generator τ_i satisfies the algebra $[\tau_j, \tau_k] = \epsilon_{ijk}\tau_i$.

The Hamiltonian method leading to the definition of the momentum in (2) is the cause of the noncovariant formalism (E_i^μ equals identically zero). In this formalism, the Poisson algebra of the Ashtekar-Barbero variables is canonical,

$$\big\{A_a^i(x), E_j^b(y)\big\}_{q,p} = -\frac{\gamma\kappa}{2}\delta_a^b \delta_j^i \delta^3(x-y)\,, \qquad \big\{A_a^i(x), A_a^j(y)\big\}_{q,p} = \big\{E_i^a(x), E_j^b(y)\big\}_{q,p} = 0\,, \tag{3}$$

where the brackets have been derived regarding the metric field canonical pair, q_{ab} and p^{ab} [19].

The canonicity of the gravitational variables suggests postulating the Heisenberg-DeWitt [23,24] form of the operators,

$$\hat{A}_a^i|\ldots\rangle = A_a^i|\ldots\rangle\,, \qquad \hat{E}_i^a|\ldots\rangle = -i\hbar\frac{\delta}{\delta A_a^i}|\ldots\rangle\,. \tag{4}$$

At each point, they should act on the states of a Hilbert space. However, to preserve SE description at the quantum level, the field operators space cannot be defined as a Fock space. The solution to this problem has been found by using the metric-independent Ashtekar-Lewandowski [25,26] measure on a gauge-invariant lattice.

Classically, the lattice is embedded in the spatial sector of the four-dimensional spacetime-representing manifold by defining the holonomy

$$h_l^{-1} := \mathcal{P}\exp\big(-\theta_l^i \tau_i\big)\,, \qquad \theta_l^i = \int_0^l ds\, \dot{\ell}^a(s)\, A_a^i\big(\ell(s)\big)\,, \tag{5}$$

which is the functional of field A_a^i. In LQG, the continuous-to-discrete transition from the distribution of this field into the distribution of the links-located holonomies is determined by reversing the expansion

$$h_l^{\mp 1}[A] = \mathbb{1} \mp l A_l + \frac{l^2}{2} A_l A_l \mp \frac{l^2}{2} \partial_l A_l + \mathcal{O}\big(l^3\big)\,. \tag{6}$$

This expansion along short links is the inverse exponential map from the representation to the group element. However, by neglecting the quadratic order terms, as in LQG [2–4], one does not obtain the map, which guarantees the correct construction of the gauge field representations on a Hilbert space [14]. Moreover, the expansion in (6) does not necessarily provide the Wigner-symmetric representation of operators [9], as has been conversely suggested in the states space construction for LQG [27].

To ensure the rigorous transfer of the classical gauge symmetry to quantum states defined as representations of the holonomy, this quantity must be expanded around the unit element of SU(2),

$$h_l^{\mp 1}[\theta] = \mathbb{1} \mp \theta_l + \frac{1}{2}\theta_l \theta_l \pm \frac{1}{2}[\theta_l, \theta_l] + \mathcal{O}\big(\theta_l^3\big)\,. \tag{7}$$

The last term expresses Lie brackets of symmetric quantities, thus it vanishes. The short links and nearly linear links approximations of the parameter θ_l^i in (5) lead to the following expansion:

$$\theta_l = \frac{l}{2}\big(A_l(v) + A_l(v{+}l)\big) + \mathcal{O}\big(l^3\big)\,. \tag{8}$$

It is worth noting that this result coincides with the expansion in (6), in which the derivative becomes approximated by a difference.

The postulates of short links and nearly linear links are reasonable assumptions imposed on the lattice's structure. By removing the latter assumption, one could not introduce a consistent, i.e. up to the quadratic order in the regulator, relation between holonomy and connection (see (10)). By strengthening this assumption to linear links (and possibly weakening the former one into the inequalities $0 \leq l < 1$), one will simplify the theory a lot. In this latter case, the lattice will become piecewise linear with holonomies of constant connections $\bar{A}_l$ at each piece (along each link). As a result, the operators $\hat{\bar{A}}_l$ (constant along links) will automatically become the Wigner operators [10], because the parameters in (7) will equal $\theta_l = l\bar{A}_l$. Moreover, all the quantities related to different first-class constraints will decouple on a piecewise linear lattice [12]. Furthermore, the domain of the Hamiltonian constraint will become separable [11].

The choice of the piecewise linear lattice, suggested by the Hilbert space construction, does not modify the algebra of the lattice-smeared gravitational field representatives, known as the holonomy-flux algebra. The lattice regularization of the gravitational momentum is introduced via the following identity:

$$\sigma |E|^{-\frac{1}{2}} \epsilon_{ijk} E_j^b E_k^c = \frac{4}{\gamma\kappa} \epsilon^{abc} \big\{ \mathbf{V}(R), A_a^i \big\}\,, \tag{9}$$

cf. [2,4], where the value of $\sigma = \pm 1$ depends on the sign of the determinant of E^a_i, namely E. The volume $\mathbf{V}(R)$ of the region R can be approximated by the functional of the momenta fluxes. By using the original short links expansion in (6), the gravitational field becomes lattice regularized via the expression

$$h_l\langle v\rangle - h_l^{-1}\langle v\rangle = l\big(A_l(v) + A_l(v{+}l)\big) + \mathcal{O}\big(l^3\big) = 2l\bar{A}_l + \mathcal{O}\big(l^3\big)\,, \tag{10}$$

cf. [11]. In the last equality, the constancy condition $A_l(v) = A_l(v{+}l)$ is assumed, where v and $v{+}l$ are the endpoints of the link l.

Independently of the selection of the analytical or linear category of links of the gauge-invariant lattice, the first-class system of constraints restricts the action of fluxes to the form, which does not mix the constraints application results [10]. The Gauss constraint, which implements the internal symmetry of the A_l field on the lattice, introduces appropriate projectors into the invariant representations at the endpoints of links [28]. Consequently, to preserve the invariance between these endpoints regarding the algebra of the variables in another first-class constraint, namely the Hamiltonian constraint (see Fig. 1), the following modification of the flux [12] is needed:

$$\bar{G}^l_i(R) := \frac{1}{l}\int_l ds\,\dot{l}(s)\,g^{-1}(l(s))\int_{S^{l(s)}} \varkappa_a E^a_i(l(s))\,g(l(s))\,, \qquad g, g^{-1} \in \mathrm{SU}(2)\,. \tag{11}$$

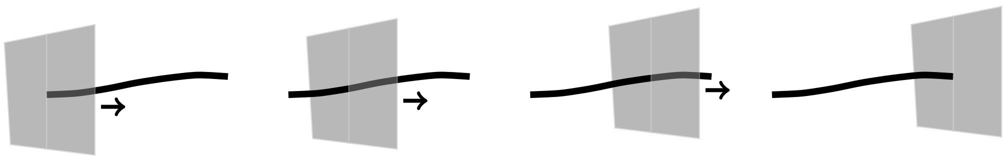

Figure 1. Flux probability distribution between the endpoints of a link

The improved representation of the flux through the surface S^l in (11) ($\varkappa_a$ is normal to this surface) is its continues distribution along the link l, orthogonal to S^l, in the region indicated by the volume $\mathbf{V}(R) = S^l \wedge l$. The gauge-invariance is implemented by the extra pair of the inverse SU(2) elements. Then, verifying the algebra of the improved representations of the lattice-smeared Ashtekar-Barbero variables in (10) and (11), one finds

$$\left\{\left(h_l - h_l^{-1}\right), \bar{G}^l_i(R)\right\} = \frac{1}{2}\left(\tau^i\left(h_l - h_l^{-1}\right) + \left(h_l - h_l^{-1}\right)\tau^i\right), \tag{12}$$

cf. [12]. Remarkably, the $\mathfrak{su}(2)$ generators did not appear between the link's endpoints, i.e. the holonomies remained undivided along the link. The appearance of these generators does not change the implementation of the gauge invariance, which has been located at the endpoints and for any internal direction. Moreover, the result in (12) is the same regarding choosing either an analytical or a linear link.

3. Results and Discussion

Independently of the selection of a piecewise analytical or linear lattice, the regularization of connections by holonomies in (10) leads to the discrete structure of the Hamiltonian constraint. This object takes the form analogous to the following expression:

$$-\frac{2\sigma}{\gamma\kappa^2}\sum_v N\langle v\rangle\,\epsilon^{pqr}\,\mathrm{tr}\Big[\Big(h_{qr}\langle v\rangle - h_{qr}^{-1}\langle v\rangle\Big)\Big\{\mathbf{V}(R\langle v\rangle), \Big(h_p\langle v\rangle - h_p^{-1}\langle v\rangle\Big)\Big\}\Big]\,. \tag{13}$$

This particular expression corresponds to the piecewise linear lattice embedded in the spatial sector of spacetime by identifying the the links with the edges in the quadrilaterally hexahedral tessellation of this spatial sector [10]. The complete formula of the Hamiltonian constraint is the sum of the expression in (13), known as the Euclidean term and the so-called Lorentzian term. The latter one can be derived from the Euclidean term by using the identity (introduced in [1,2]) based on the boost sector ${\Gamma^i}_{0a}$ of the connection in (2). If one selected a different discretization, for instance, a triangularization, one would not be able to replace the integral in the continuous formulation of the Hamiltonian with the equivalent Riemann sum,

$$\int d^3x\, f(x) = \lim_{\bar{l}\to 0} \sum_v f(R\langle v\rangle)\, \bar{l}^3\langle v\rangle\,. \tag{14}$$

It is also worth emphasizing that the separable structure of the regions $R\langle v\rangle$ with volumes $\bar{l}^3\langle v\rangle$ is determined by the diffeomorphism symmetry between these regions. This symmetry is implemented at their boundaries and ensures the preservation of the orientations between the bounded regions. The introduction of the separable structure is possible by the precise indication of the boundaries, which are the six faces of each quadrilateral hexahedron. It is worth mentioning that this structure is known as a fiber sum. The just described restrictions concern the construction of the lattice, in which nodes v label the basic separable regions in (13). These restrictions are also the reason that prevents indicating the analogous classical discrete expression of the Hamiltonian constraint on a piecewise analytical tetrahedral lattice, by using the original methods of LQG [1–4].

The classical discrete formulation of the Hamiltonian constraint on a gauge-invariant lattice is a significant result. It allows for the application of the standard canonical quantization procedure from QFT. This application will lead to the quantum theory with a known method of derivation of the (semi)classical limit. Moreover, by imposing the phase space reduction on the lattice system with gauge symmetries, one can be sure that the resulting model will preserve the same reduced symmetries after the quantization [17]. This phase space reduction procedure allows to formulate the corresponding theory of lattice cosmology, which is the cosmological limit of the lattice gravity and will remain this limit at the quantum level. It is worth noting that this relation does not exist [15] regarding LQG and the popular but independent effective cosmological model, called loop quantum cosmology (LQC) [6,7,29,30].

The phase space reduction is a technique in which appropriate gauge-fixing conditions are added *ad hoc* to a first-class constraints system. These conditions, by forming a second-class system with the selected first-class constraints, have to fix the appropriate symmetry completely [31]. Concerning lattice gravity, the conditions, which lead to the cosmological model, fix the internal symmetry to the Abelian case and the diffeomorphism invariance to the spatial translations. By defining these gauge-fixing conditions in the way that they do not Poisson-commute with the spatial and internal symmetries, one significantly simplifies the system. It becomes describable by variables constant along the gauge symmetries-related spatial and internal directions. Moreover, by identifying these directions, the constraints generating gauge symmetries vanish identically. The choice of the gauge fixing conditions [15] has been inspired by the postulated form of the anisotropic LQC variables [6,30]. Consequently, by deriving the so-normalized Abelian, constant, and diagonal variables

$$\begin{aligned}
&\mathtt{A}(l_p) \to c_i(l_{(i)})[h]\tau_i := \mathbb{L}_0 \varepsilon_{(i)}\langle v\rangle A^i_p[h_{(p)}]\, {}^0\!e^p_{(i)}\tau_i\,, \qquad h_i\langle v\rangle[c] - h_i^{-1}\langle v\rangle[c] = 2c_i(l^{(i)})\,\tau_{(i)}\,,\\
&\mathtt{E}(F^p) \to p^i(F^{(i)})[f]\tau_i := \frac{\varepsilon_{(i)}\langle v\rangle}{\mathbb{L}_0^2\,\bar{\varepsilon}^3\langle v\rangle} E^p_i[f^{(p)}]\,\frac{{}^0\!e^{(i)}_p}{\sqrt{{}^0\!q}}\tau_i\,, \qquad f(F^i\langle v\rangle)\,[p] = f^{(i)}(F^i\langle v\rangle)\,[p]\,\tau_{(i)} = p^i(F^{(i)})\,\tau_{(i)}\,,
\end{aligned} \tag{15}$$

one can expect that the resulting model will describe the cosmological solution [15]. What should be emphasized regarding these variables, which solve the gauge-fixing conditions on the lattice, is that the right-hand side of the formulas in (15) expresses the discrete quantities. The objects ${}^0\!e^p_i$, ${}^0\!e^i_p$ are the diagonal constant unit matrices and the square root of the diagonal constant metric tensor determinant is given by the expression $\sqrt{{}^0\!q} := \epsilon_{ijk}\epsilon^{pqr}\,{}^0\!e^i_p\,{}^0\!e^j_q\,{}^0\!e^k_r/3!\,$.

It is easy to see that in the resulting Hamiltonian there will be no loop contribution h_{qr} that was present in (13). This object was the result of the regularization of the gravitational connection curvature F_{qr}. In the Abelian case, the curvature takes the form $F_{qr}^{\mathrm{U}(1)} = \partial_q A_r - \partial_r A_q$ and it vanishes in the case of the globally spatially constant connection in (15). Moreover, this constancy leads to the vanishing of the Lorentzian term in the Hamiltonian (see the discussion below (13)). As a result, only the expression in (13) remains, becoming restricted to a cuboidal (rectangularly hexahedral) lattice and the variables in (15). See [16] for details.

For a reader familiar with LQC, it may be worth emphasizing that the algebra of the lattice-reduced variables in (15) is different than the algebra of the postulated variables of the former effective model. In the case of LQC, the Poisson brackets of the postulated constant and diagonal variables read $\{c_i, p^j\} = -\frac{\gamma\kappa}{2}\delta_i^j$ [6,30]. By imposing the reduction on the lattice, the canonical variables are $h_i\langle v\rangle[c] - h_i^{-1}\langle v\rangle[c]$ and $f(F^i\langle v\rangle)[p]$. The corresponding Poisson brackets are

$$\{c_i(l\langle v\rangle)[h], p^j(F\langle v'\rangle)[f]\} = \mathrm{tr}\Big(\Big\{f(F^{(i)}\langle v'\rangle)[p], h_{(i)}\langle v\rangle[c] - h_{(i)}^{-1}\langle v\rangle[c]\Big\}\Big)\delta_i^j = -\gamma\kappa\, c_{(i)}(l^{(i)})\,\tau_{(i)}\delta_{l\langle v\rangle l\langle v'\rangle}\delta_i^j\,. \quad (16)$$

One can check that they agree with the Abelian, constant, and diagonal limit of the algebra in (12).

4. Conclusions

The total Hamiltonian of the cosmologically-reduced lattice gravity [16] consists of a single constraint

$$H(\Gamma) = -\frac{1}{\gamma^2\kappa}\sum_{R_{c\langle v\rangle}} N(R_{c\langle v\rangle})\,\frac{\bar{p}^i(R)\,\bar{p}^j(R)}{(|\bar{p}^1(R)\,\bar{p}^2(R)\,\bar{p}^3(R)|)^{\frac{1}{2}}}\big(\bar{c}_i(R)\,\bar{c}_j(R) - \bar{c}_j(R)\,\bar{c}_i(R)\big)\,. \quad (17)$$

It is the Hamiltonian constraint, which takes the form of a sum over elementary regions that are going to be called cells. The cell-related variables turn out to be the following sums:

$$\bar{c}_i(R_{c\langle v\rangle})[h] = \frac{1}{4}\sum_{v\in F^{(i)}\langle v\rangle}\mathrm{tr}\Big(\tau^{(i)}\big(h_i^{-1}\langle v\rangle[c] - h_i\langle v\rangle[c]\big)\Big)\,, \qquad \bar{p}_i(R_{c\langle v\rangle})[f] = \frac{1}{2}\sum_{v\in l_{(i)}\langle v\rangle} f^{(i)}(F^i\langle v\rangle)[p]\,. \quad (18)$$

The distribution of the cosmological holonomy oriented toward an i-th direction in a given cell is the sum of the reduced holonomies along four links that are the edges of this cuboidal cell, which are parallel to the i-th direction. The cosmological flux is simply the reduced flux through the opposite faces of a given cuboidal cell.

It should be emphasized that this result is fundamental, i.e. it has been derived by using well-established methods of gauge fixing in QFT. No additional approximations, neither perturbations were introduced. Moreover, this result is finite, and contrary to LQC, one does not need to introduce any *ad hoc* cut-off. And yet, regardless of the serious and rigorous reduction, the result still describes anisotropies (and trivial inhomogeneities in terms of the lapse function). Furthermore, the anisotropic framework is non-trivially constrained by the global cuboidal structure of the lattice. This structure has to remain globally cuboidal for each action of the operators after the canonical quantization. However, the deviations from the cubic structure can be different for different values of the time.

The lattice-reduced system allows investigating the evolution of quantum cosmology *à la* quantum mechanics. Moreover, this evolution looks to be expressible in terms of transition amplitudes between different cuboidal configurations. Furthermore, by concerning these configurations as different deviations from the cubic structure, this system appears to be a description of a fundamental process of self-isotropization. Finally, it seems that if the size of the lattice expands during this process, the anisotropies will remain frozen within the history at large scales. Does it not look like a postulated evolution of the early universe?

It should be indicated that the last two comments are not supported with a rigorous calculation. They are only the author's predictions based on the structure of the result in (17) and some early heuristic checks. A detailed numerical investigation is needed to verify whether these expectations are right and if their details agree with cosmological observations.

Acknowledgments: This work was partially supported by the National Natural Science Foundation of China grants Nos. 11675145 and 11975203.

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