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De Sitter Solutions In Models With The Gauss-Bonnet Term

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Abstract: De Sitter solutions play an important role in cosmology because the knowledge of unstable de Sitter solutions can be useful to describe inflation, whereas stable de Sitter solutions are often used in models of late-time acceleration of the Universe. The models the Gauss-Bonnet term are actively used both as inflationary models and as dark energy models. To modify the Einstein equations one can add a nonlinear function of the Gauss-Bonnet term or a function of the scalar field multiplied on the Gauss-Bonnet term. The effective potential method essentially simplifies the search and stability analysis of de Sitter solutions, because the stable de Sitter solutions correspond to minima of the effective potential.

Keywords: Einstein-Gauss-Bonnet gravity; De Sitter solution; stability

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1. Introduction

It is well-known that one can add the Gauss-Bonnet term to the Hilbert-Einstein Lagrangian of the General Relativity and it does not change the equations of motion. On the other hand, this term multiplied by some nonconstant function of a scalar field modifies the equations of motion. Also, models with a non-linear function of the Gauss-Bonnet term can be rewritten in the equivalent form that includes a scalar field without kinetic term.

The cosmological models with the Gauss-Bonnet term are motivated by the string theory [1–8] and are actively used for describing of both the early Universe evolution [9–24] and the current dark energy dominated epoch [5–7,25–30].

Note that these both studies of the Universe evolution are characterized by the quasi de Sitter accelerated expansion of the Universe. So, it is important to have an effective method for the searing of de Sitter solutions and the study of their stability. For the Gauss-Bonnet model with the standard scalar field, such method has been proposed in [31]. It is a generalization of the effective potential method [32,33]. In this paper, we generalize this method on model with nonlinear functions of the Gauss-Bonnet term. We also consider the case of a phantom scalar field and show that in this case the situation is more difficult.

2. Models the Gauss-Bonnet term

Let us consider the model with the Gauss–Bonnet term described by the following action:

$$S = \int d^4x\sqrt{-g}\Big[UR - \frac{c}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V - F\mathcal{G}\Big], \tag{1}$$

where the functions $U(\phi)$, $V(\phi)$, and $F(\phi)$ are double differentiable ones, c is a constant, R is the Ricci scalar and $\mathcal{G}$ is the Gauss–Bonnet term,

$$\mathcal{G} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}.$$

Note that the action

$$S = \int d^4x \sqrt{-g} W(\mathcal{G}), \tag{2}$$

where $W(\mathcal{G})$ is a double differentiable function, can be rewritten in the following form [8, 27]:

$$S = \int d^4x \sqrt{-g} \left[W'(\phi)(\mathcal{G} - \phi) + W(\phi) \right], \tag{3}$$

where a prime denotes the derivatives with respect to ϕ. Varying action (3) over ϕ, one gets $\phi = \mathcal{G}$ and the initial $W(\mathcal{G})$ model. Therefore, action (1) with $c = 0$ describes $W(\mathcal{G})$ models.

In the spatially flat Friedmann–Lemaître–Robertson–Walker metric with

$$ds^2 = -dt^2 + a^2(t)\left(dx_1^2 + dx_2^2 + dx_3^2\right), \tag{4}$$

one gets the following evolution equations:

$$6H^2U + 6HU'\dot{\phi} = \frac{c}{2}\dot{\phi}^2 + V + 24H^3F'\dot{\phi}, \tag{5}$$

$$4(U - 4H\dot{F})\dot{H} = -c\dot{\phi}^2 - 2\ddot{U} + 2H\dot{U} + 8H^2(\ddot{F} - H\dot{F}), \tag{6}$$

$$c\ddot{\phi} + 3cH\dot{\phi} - 6\left(\dot{H} + 2H^2\right)U' + V' + 24H^2F'\left(\dot{H} + H^2\right) = 0, \tag{7}$$

where $H = \dot{a}/a$ is the Hubble parameter, dots and primes denote the derivatives with respect to the cosmic time and the scalar field ϕ, respectively. At $c = 1$ these equations have been investigated in many papers (see, for example [11,31]).

To find de Sitter solutions with a constant ϕ in the model (2) we substitute $\phi = \phi_{dS}$ and $H = H_{dS}$ into Eqs. (5) and (7). A de Sitter solution does not depend on the value of c, so we obtain the same results as in the case $c = 1$ considered in [31]:

$$H_{dS}^2 = \frac{V_{dS}}{6U_{dS}} \tag{8}$$

and

$$F'_{dS} = \frac{3U_{dS}(2U'_{dS}V_{dS} - V'_{dS}U_{dS})}{2V_{dS}^2}, \tag{9}$$

where $A_{dS} \equiv A(\phi_{dS})$ for any function A. Therefore, for arbitrary functions $U(\phi)$ and $V(\phi)$ with $V_{dS}U_{dS} > 0$, we can choose $F(\phi)$ such that the corresponding point becomes a de Sitter solution, with the Hubble parameter defined by Eq. (8). We always choose that $H_{dS} > 0$.

3. Stability of de Sitter solutions

To analyze the stability of a de Sitter solution we transform Eqs. (6)–(7) to the following dynamical system:

$$\begin{aligned}
\dot{\phi} =& \psi, \\
\dot{\psi} =& \frac{1}{2(\tilde{B} - 4cF'H\psi)}\Big\{2H\left[3B + 4F'V' - 6U'^2 - 6cU\right]\psi - 2\frac{V^2}{U}X \\
& + \left[12H^2\left[(2U'' + 3c)F' + 2U'F''\right] - 96F'F''H^4 - 3(2U'' + c)U'\right]\psi^2\Big\}, \\
\dot{H} =& \frac{1}{4(\tilde{B} - 4cF'H\psi)}\Big\{8c\left(U' - 4F'H^2\right)H\psi \\
& - 2\frac{V^2}{U^2}\left(4F'H^2 - U'\right)X + \left(8F''H^2 - 2U'' - c\right)c\psi^2\Big\},
\end{aligned} \tag{10}$$

where

$$\tilde{B} = 3\left(4H^2F' - U'\right)^2 + cU, \tag{11}$$

$$X = \frac{U^2}{V^2}\left[24H^4F' - 12H^2U' + V'\right]. \tag{12}$$

In the case $c = 0$, the last equation is essentially simplified:

$$\dot{H} = \frac{24H^4F' - 12H^2U' + V'}{6(U' - 4H^2F')}. \tag{13}$$

At a de Sitter point system (10) is

$$\dot{\phi} = 0, \quad \dot{\psi} = 0, \quad \dot{H} = 0,$$

that corresponds to $X_{dS} = 0$.

In Ref. [31], the effective potential has been proposed for models with the Gauss-Bonnet term:

$$V_{eff} = -\frac{U^2}{V} + \frac{2}{3}F. \tag{14}$$

Using Eq. (8), we obtain

$$X_{dS} = \frac{2}{3}F'_{dS} - 2\frac{U'_{dS}U_{dS}}{V_{dS}} + \frac{V'_{dS}U^2_{dS}}{V^2_{dS}} = V'_{eff}(\phi_{dS}) = 0, \tag{15}$$

therefore, de Sitter solutions correspond to extremum points of the effective potential V_{eff}.

To investigate the Lyapunov stability of a de Sitter solution we use the following expansions:

$$H(t) = H_{dS} + \varepsilon H_1(t) \quad \phi(t) = \phi_{dS} + \varepsilon\phi_1(t), \quad \psi(t) = \varepsilon\psi_1(t), \tag{16}$$

where ε is a small parameter. Therefore,

$$X = \varepsilon(X_{,H}H_1 + X_{,\phi}\phi_1) + \mathcal{O}(\varepsilon^2), \tag{17}$$

where

$$X_{,H} = \left.\frac{\partial X}{\partial H}\right|_{\phi=\phi_{dS}} = \frac{4\sqrt{6}}{V_{dS}^{5/2}}U_{dS}^{3/2}(U'_{dS}V_{dS} - V'_{dS}U_{dS}),$$

$$X_{,\phi} = \left.\frac{\partial X}{\partial \phi}\right|_{\phi=\phi_{dS}} = \frac{1}{V^2_{dS}}\left(\frac{2}{3}F''_{dS}V^2_{dS} - 2U''_{dS}U_{dS}V_{dS} + V''_{dS}U^2_{dS}\right).$$

The functions $H_1(t)$, $\phi_1(t)$, and $\psi_1(t)$ are connected by Eq. (5):

$$H_1(t) = \frac{V'_{dS}U_{dS} - U'_{dS}V_{dS}}{2U_{dS}V_{dS}}(H_{dS}\phi_1(t) - \psi_1(t)). \tag{18}$$

This expression does not depend on the value of c and coincide with the corresponding expression obtained in Ref. [31].

Substituting (16), (17), and (18) into Eq. (10) in the first order of ε, we obtain the following system of two linear differential equations:

$$\dot{\phi}_1 = \psi_1, \tag{19}$$

$$\dot{\psi}_1 = -\frac{\left[2F''_{dS}V^3_{dS} - 6U''_{dS}U_{dS}V^2_{dS} + 3V''_{dS}U^2_{dS}V_{dS} - 6\left(U'_{dS}V_{dS} - V'_{dS}U_{dS}\right)^2\right]}{3U_{dS}V_{dS}B_{dS}}\phi_1 - \frac{\sqrt{6U_{dS}V_{dS}}}{2U_{dS}}\psi_1, \tag{20}$$

where

$$B_{dS} = \frac{3}{V_{dS}^2}\left(V'_{dS}U_{dS} - U'_{dS}V_{dS}\right)^2 + cU_{dS}. \tag{21}$$

This system can be rewritten in the matrix form:

$$\begin{pmatrix} \dot{\phi}_1 \\ \dot{\psi}_1 \end{pmatrix} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{21} \\ \tilde{A}_{12} & \tilde{A}_{22} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \psi_1 \end{pmatrix} \tag{22}$$

where

$$\tilde{A} = \left\| \begin{array}{cc} 0, & 1 \\ -\frac{V_{dS}^2 V''_{eff}(\phi_{dS})}{U_{dS}B_{dS}}, & -3H_{dS} \end{array} \right\|$$

The solution of system (22) has the following form

$$\phi_1 = c_{11}\mathrm{e}^{-\lambda_- t} + c_{21}\mathrm{e}^{-\lambda_+ t}, \tag{23}$$

$$\psi_1 = c_{21}\mathrm{e}^{-\lambda_- t} + c_{22}\mathrm{e}^{-\lambda_+ t}, \tag{24}$$

where c_{ij} are some constants. Solving the characteristic equation:

$$\det(\tilde{A} - \lambda \cdot I) = \lambda^2 - 3H_{dS}\lambda + \frac{V_{dS}^2 V''_{eff}(\phi_{dS})}{U_{dS}B_{dS}} = 0, \tag{25}$$

we get the following roots:

$$\lambda_\pm = -\frac{3}{2}H_{dS} \pm \sqrt{\frac{9}{4}H_{dS}^2 - \frac{V_{dS}^2}{U_{dS}B_{dS}}V''_{eff}(\phi_{dS})}\,. \tag{26}$$

A de Sitter solution is stable if real parts of both λ_- and λ_+ are negative. We consider the case $H_{dS} = \sqrt{\frac{V}{6U}} > 0$, hence, $\Re e(\lambda_-) < 0$.

In the case of a positive U_{dS}, we see that $B_{dS} > 0$ for $c \geqslant 0$ and the condition $\Re e(\lambda_+) < 0$ is equivalent to $V''_{eff}(\phi_{dS}) > 0$. In the cases $c > 0$ and $c = 0$, a de Sitter solution is stable if $V''_{eff}(\phi_{dS}) > 0$ and unstable if $V''_{eff}(\phi_{dS}) < 0$.

In the case $c < 0$, we see that B_{dS} can be negative. So, in this case de Sitter solution is stable if the $V''_{eff}(\phi_{dS})B_{dS} > 0$. So, the main result of Ref. [31] can be generalized on the case $c = 0$ without any correction, whereas the condition should be change to $V''_{eff}(\phi_{dS})B_{dS} > 0$ in the case of $c < 0$ that corresponds to a phantom scalar field ϕ.

4. Conclusions

In this paper, we consider de Sitter solutions in models with the Gauss-Bonnet term, including $W(\mathcal{G})$ models. We show that the effective potential proposed [31] for model with the Gauss-Bonnet term multiplied on a function of the scalar field can be used in $W(\mathcal{G})$ models as well. To find de Sitter solutions in some $W(\mathcal{G})$ model, we rewrite the action of this model in the form (3) and construct the corresponding effective potential V_{eff}. A stable de Sitter solutions corresponds $V''_{eff}(\phi_{dS}) > 0$, where the values of the scalar field at de Sitter point ϕ_{dS} is determined by the condition $V'_{eff}(\phi_{dS}) = 0$.

Note that the effective potential is useful tools for construction of inflationary scenarios in the models with the Gauss-Bonnet term multiplied to a function of the scalar field [24]. We plan to generalize this approach to inflationary scenarios in $W(\mathcal{G})$ models.

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