De Sitter solutions in models with the Gauss–Bonnet term

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based on

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STRING THEORY MOTIVATED GRAVITY

The Gauss–Bonnet models are motivated by α' corrections in string theories. The most general Lagrangian density at the next to leading order in the parameter α' reads[1]:

$$L_{string} = -\frac{\lambda}{2}\alpha'\xi(\phi)\left[c_1\mathcal{G} + c_2 G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + c_3\Box\phi\phi^{;\mu}\phi_{;\mu} + c_4(\phi^{;\mu}\phi_{;\mu})^2\right],$$

where

- $\mathcal{G}$ is the Gauss–Bonnet term:

$$\mathcal{G} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta},$$

- $G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R$ is the Einstein tensor,
- $\alpha' = \lambda_s^2$, where λ_s is the fundamental string length scale;
- c_i are constants (we will consider the case $c_k = 0$, $k = 2, 3, 4$);
- λ is an additional parameter allowing for different species of string theories, $\lambda = -1/4$ for the Bosonic string and $\lambda = -1/8$ for Heterotic string respectively.

[1] D.J. Gross and J.H. Sloan, Nucl. Phys. B **291** (1987) 41;
R.R. Metsaev and A.A. Tseytlin, Nucl. Phys. B **293** (1987) 385.

MODELS WITH THE GAUSS–BONNET TERM

There are two basic motivations which lead cosmologists to modify gravity.

The first one is an attempt to connect gravity with quantum physics, at least in a perturbative way, by including quantum correction terms to Einstein's equations.

The second one is an interest to describe the Universe accelerated expansion in a natural way, without the dark energy.

FOUR EPOCHS

Reliable astronomical data support the existence of four distinct epochs of the Universe global evolution:

- an inflation,
- a radiation dominated era,
- a matter dominated era,
- the present dark energy epoch.

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Initial inflation and dark energy domination are both characterized by an accelerated expansion of the Universe with almost constant Hubble parameter H (quasi de Sitter solution).
The other epochs of the Universe evolution are described by power-law solutions with $H = J/t$, where J is a positive constant.
In General Relativity, power-law solutions with $H = J/t$ correspond to models with a perfect fluid whose EoS parameter reads
$w_{\rm m} = -1 + 2/(3J)$.
The radiation dominated epoch corresponds to solutions with $J = 1/2$, whereas the matter dominated one corresponds to $J = 2/3$.

INFLATIONARY MODELS

The perturbation theory for such types of models has been developed in
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Z.K. Guo and D.J. Schwarz, Phys. Rev. D **81**, 123520 (2010) [arXiv:1001.1897]
A. De Felice, S. Tsujikawa, J. Elliston and R. Tavakol, JCAP **08** (2011) 021 [arXiv:1105.4685]
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DARK ENERGY MODELS

Models the Gauss–Bonnet term successfully generate a dark energy era.

G. Calcagni, S. Tsujikawa and M. Sami, *Dark energy and cosmological solutions in second-order string gravity*, Class. Quant. Grav. **22** (2005) 3977 [arXiv:hep-th/0505193]

S. Tsujikawa and M. Sami, *String-inspired cosmology: Late time transition from scaling matter era to dark energy universe caused by a Gauss-Bonnet coupling*, J. Cosmol. Astropart. Phys. **0701** (2007) 006 [arXiv:hep-th/0608178]

S. Nojiri, S.D. Odintsov and M. Sasaki, *Gauss-Bonnet dark energy*, Phys. Rev. D **71** (2005) 123509 [arXiv:hep-th/0504052]

S. Capozziello, A.N. Makarenko and S.D. Odintsov, Gauss-Bonnet dark energy by Lagrange multipliers, Phys. Rev. D **87** (2013) 084037 [arXiv:1302.0093].

M. Benetti, S. Santos da Costa, S. Capozziello, J. S. Alcaniz and M. De Laurentis, Observational constraints on Gauss–Bonnet cosmology, Int. J. Mod. Phys. D **27** (2018) 1850084 [arXiv:1803.00895].

THE EINSTEIN–GAUSS–BONNET GRAVITY

The model with the Gauss–Bonnet term is described by the action:

$$S = \int d^4x\sqrt{-g}\left(U(\phi)R - \frac{c}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) - F(\phi)\mathcal{G}\right), \qquad (1)$$

where U, V, and F are differentiable functions and c is a constant.
Let us consider the action

$$S_{\mathcal{G}} = \int d^4x\sqrt{-g}\,W(\mathcal{G}), \qquad (2)$$

where W is a differentiable function. Action $S_{\mathcal{G}}$ can be linearized with respect to the Gauss–Bonnet term, by adding a scalar field in the action[2]. Introducing a field ϕ without kinetic term, we obtain the following action:

$$S_{\mathcal{G}\phi} = \int d^4x\sqrt{-g}\left[\left[\frac{dW}{d\phi}(\mathcal{G}-\phi) + W(\phi)\right]\right].$$

Varying over ϕ, one gets $\phi = \mathcal{G}$ and reconstruct $S_{\mathcal{G}}$. So, $S_{\mathcal{G}}$ can be written as action (3) with $c = 0$.

[2] G. Cognola, E. Elizalde, S. Nojiri, S.D. Odintsov and S. Zerbini, Phys. Rev. D **73** (2006) 084007 [arXiv:hep-th/0601008]

MODELS WITH THE GAUSS–BONNET TERM

$$S=\int d^4x\sqrt{-g}\left(U(\phi)R-\frac{c}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-V(\phi)-F(\phi)\mathcal{G}\right). \qquad (3)$$

In the spatially flat FLRW universe with the interval

$$ds^2=\,-\,dt^2+a^2(t)\left(dx_1^2+dx_2^2+dx_3^2\right),$$

one gets the following equations

$$6H^2U+6HU'\dot\phi=\frac{c}{2}\dot\phi^2+V+24H^3F'\dot\phi, \qquad (4)$$

$$4\left(U-4H\dot F\right)\dot H=\,-\,c\dot\phi^2-2\ddot U+2H\dot U+8H^2\left(\ddot F-H\dot F\right), \qquad (5)$$

$$c\ddot\phi+3cH\dot\phi-6\left(\dot H+2H^2\right)U'+V'+24H^2F'\left(\dot H+H^2\right)=0, \qquad (6)$$

where $H=\dot a/a$ is the Hubble parameter, primes mean the derivatives with respect to ϕ, dots mean the derivatives with respect to t.

DE SITTER SOLUTIONS

Let us find de Sitter solutions in the model with the Gauss–Bonnet term. It would be convenient, if all the necessary information on the existence and stability of de Sitter solutions is obtained from a single combination of functions U, V, and ξ dubbed effective potential V_{eff}.

We restrict ourselves to de Sitter solutions with a constant ϕ.

Substituting $\phi = \phi_{dS}$ and $H = H_{dS}$ into Eqs. (4) and (6), we get:

- The equation for the Hubble parameter at the de Sitter point is the same as in the corresponding model without the Gauss–Bonnet term:

$$H_{dS}^2 = \frac{V_{dS}}{6U_{dS}}. \tag{7}$$

- For arbitrary functions U and V with $VU > 0$, we can choose $F(\phi)$ such that the corresponding point becomes a de Sitter solution with the Hubble parameter defined by (7). The value of $F'(\phi_{dS})$ is

$$F'_{dS} = \frac{3U_{dS}(2U'_{dS}V_{dS} - V'_{dS}U_{dS})}{2V_{dS}^2}, \tag{8}$$

where $A_{dS} \equiv A(\phi_{dS})$ for any function A.

THE EFFECTIVE POTENTIAL

It would be convenient to obtain position and stability of de Sitter solutions using only one combination of three functions: U, V, and ξ. To get this combination (the effective potential) we cast Eqs. (6) and (5) as a dynamical system:

$$\begin{aligned}
\dot{\phi} =& \psi, \\
\dot{\psi} =& \frac{1}{2\left(\tilde{B} - 4cF'H\psi\right)} \left\{2H\left[3B + 4F'V' - 6U'^2 - 6cU\right]\psi - 2\frac{V^2}{U}X \right. \\
& \left. + \left[12H^2\left[(2U'' + 3c)F' + 2U'F''\right] - 96F'F''H^4 - 3(2U'' + c)U'\right]\psi^2\right\}, \\
\dot{H} =& \frac{1}{4\left(\tilde{B} - 4cF'H\psi\right)} \left\{8c\left(U' - 4F'H^2\right)H\psi \right. \\
& \left. - 2\frac{V^2}{U^2}\left(4F'H^2 - U'\right)X + \left(8F''H^2 - 2U'' - c\right)c\psi^2\right\},
\end{aligned} \tag{9}$$

where

$$\tilde{B} = 3\left(4H^2F' - U'\right)^2 + cU, \qquad X = \frac{U^2}{V^2}\left[24H^4F' - 12H^2U' + V'\right].$$

We introduce the effective potential $V_{eff}(\phi)$ in the model with the Gauss–Bonnet term, such that

$$V'_{eff}(\phi_{dS}) = X(\phi_{dS}) = 0. \tag{10}$$

Indeed, let

$$V_{eff} = -\frac{U^2}{V} + \frac{1}{3}\xi. \tag{11}$$

we get

$$X(\phi_{dS}) = \frac{1}{3}\xi'_{dS} - 2\frac{U'_{dS}U_{dS}}{V_{dS}} + \frac{V'_{dS}U^2_{dS}}{V^2_{dS}} = V'_{eff}(\phi_{dS}) = 0. \tag{12}$$

De Sitter solutions correspond to extremum points of the effective potential V_{eff}.

THE LYAPUNOV STABILITY

To investigate the Lyapunov stability of a de Sitter solution we use the following expansions,

$$H(t) = H_{dS} + \varepsilon H_1(t),\ \phi(t) = \phi_{dS} + \varepsilon\phi_1(t),\ \psi(t) = \varepsilon\psi_1(t), \qquad (13)$$

where ε is a small parameter.
The functions $H_1(t)$, $\phi_1(t)$ and $\psi_1(t)$ are not independent. From Eq. (4), we obtain

$$H_1 = \frac{V'_{dS}U_{dS} - U'_{dS}V_{dS}}{2U_{dS}V_{dS}}\left(H_{dS}\phi_1 - \psi_1\right). \qquad (14)$$

Substituting (13) and (14) into (9), we get:

$$\dot{\phi}_1 = \tilde{A}_{11}\phi_1 + \tilde{A}_{12}\psi_1\,, \qquad (15)$$
$$\dot{\psi}_1 = \tilde{A}_{21}\phi_1 + \tilde{A}_{22}\psi_1\,, \qquad (16)$$

where

$$\tilde{A} = \left\| \begin{array}{cc} 0, & 1 \\ -\frac{V_{dS}^2 V''_{eff}(\phi_{dS})}{U_{dS}B_{dS}}, & -3H_{dS} \end{array} \right\|$$

Solving the characteristic equation:

$$\det(\tilde{A} - \lambda \cdot I) = \lambda^2 - 3H_{dS}\lambda + \frac{V_{dS}^2 V''_{eff}(\phi_{dS})}{U_{dS} B_{dS}} = 0, \tag{17}$$

we get the following roots:

$$\lambda_{\pm} = -\frac{3}{2}H_{dS} \pm \sqrt{\frac{9}{4}H_{dS}^2 - \frac{V_{dS}^2}{U_{dS}B_{dS}}V''_{eff}(\phi_{dS})}\,. \tag{18}$$

A de Sitter solution is stable if real parts of both λ_- and λ_+ are negative.

To get this result, we assume that $H_{dS} = \sqrt{\frac{V}{6U}} > 0$, hence, $\Re e(\lambda_-) < 0$.
In the case of a positive U_{dS}, we see that $B_{dS} > 0$ for $c \geqslant 0$. The condition $\Re e(\lambda_+) < 0$ is equivalent to $V''_{eff}(\phi_{dS}) > 0$.
In the cases $c > 0$ and $c = 0$, a de Sitter solution is stable if $V''_{eff}(\phi_{dS}) > 0$ and unstable if $V''_{eff}(\phi_{dS}) < 0$.
In the case $c < 0$, we see that B_{dS} can be negative. So, in this case de Sitter solution is stable if the $V''_{eff}(\phi_{dS})B_{dS} > 0$.
Let us consider a few examples of models with $c = 1$.

Let us consider the string theory inspired cosmological model with [3]

$$U = U_0, \quad V = \tilde{c}\mathrm{e}^{-\lambda\phi}, \quad F = \frac{\alpha}{\mu}\mathrm{e}^{\mu\phi}, \tag{19}$$

where U_0, α, $\tilde{c}$, λ, and μ are positive constants.
In this model, the effective potential is

$$V_{eff} = -\frac{U_0^2}{\tilde{c}}\mathrm{e}^{\lambda\phi} + \frac{2\alpha}{3\mu}\mathrm{e}^{\mu\phi}. \tag{20}$$

The condition $V'_{eff}(\phi_{dS}) = 0$ gives

$$\phi_{dS} = \frac{1}{\lambda-\mu}\ln\left(\frac{\alpha\tilde{c}}{3U_0^2\lambda}\right). \tag{21}$$

[3] S. Tsujikawa and M. Sami, *String-inspired cosmology: Late time transition from scaling matter era to dark energy universe caused by a Gauss–Bonnet coupling*, J. Cosmol. Astropart. Phys. **0701** (2007) 006 [arXiv:hep-th/0608178]

There exists a de Sitter solution for all $\mu \neq \lambda$. It is easy to see that $V''_{eff} = 0$ at

$$\phi_2 = \frac{1}{\lambda - \mu} \ln\left(\frac{2\alpha\tilde{c}\mu}{3U_0^2\lambda^2}\right) = \phi_{dS} - \frac{\ln(\lambda) - \ln(\mu)}{\lambda - \mu}\,, \qquad (22)$$

and $\phi_{dS} > \phi_2$ for any $\lambda \neq \mu$.

• If $\mu > \lambda$, then V''_{eff} is positive at large ϕ, so the second derivative is positive at the de Sitter point and this point is stable.

• In the opposite case, $\mu < \lambda$, $V''_{eff} < 0$ at large ϕ and the de Sitter solution is unstable.

This result of the paper
E.O. Pozdeeva, M. Sami, A.V. Toporensky, S.Yu. Vernov,
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coincides with the result obtained in
S. Tsujikawa and M. Sami, J. Cosmol. Astropart. Phys. **0701** (2007) 006 [arXiv:hep-th/0608178]
by an another method.

• We generalize this result assuming that the constants can be negative:

$$V_{eff} = c_1 \mathrm{e}^{-N_1\phi} + c_2 \mathrm{e}^{-N_2\phi}, \tag{23}$$

• The same effective potential corresponds to different choice of functions F, V, and U.
• If two of these functions are given, then we can get the third function using the given form of the effective potential.
• It is a way of constructing models with de Sitter solutions.

For example, the model ($V_{eff} = c_1 \mathrm{e}^{-N_1\phi} + c_2 \mathrm{e}^{-N_2\phi}$) with a non-minimally coupled scalar field defined by functions

$$U = U_0\left(\xi\phi^2+1\right)\mathrm{e}^{\eta_1\,\phi}, \quad \text{and} \quad V = V_0\phi^4 e^{\eta_2\phi},$$

has the effective potential given by (23) if

$$F = \frac{3}{2}\left[\frac{4U_0^2\mathrm{e}^{2\,\eta_1\phi-\eta_2\phi}}{V_0}\left(\xi+\frac{1}{\phi^2}\right)^2 + c_1\mathrm{e}^{-N_1\phi} + c_2\mathrm{e}^{-N_2\phi}\right]$$

In this model, c_i and N_i are arbitrary constants. The analysis of the second derivative of V_{eff} gives the following stability conditions:

- if $c_1 > 0$ and $c_2 > 0$, then the de Sitter solution is stable;
- if $c_1 < 0$ and $c_2 < 0$, then the de Sitter solution is unstable;
- if $c_1 > 0$ and $c_2 < 0$, then the de Sitter solution is stable at $|N_1| > |N_2|$ and unstable at $|N_1| < |N_2|$;
- if $c_1 < 0$ and $c_2 > 0$, then the de Sitter solution is stable at $|N_1| < |N_2|$ and unstable at $|N_1| > |N_2|$.

• The effective potential can be used not only to simplify the analysis of the stability of de Sitter solutions in a given model, but also to construct a new model with de Sitter solutions.

MODELS WITH $V = CU^2$

- Let us consider the case $V = CU^2$, where C is a positive constant.
- In this case, a model without the Gauss–Bonnet term transforms to a model with a constant potential in the Einstein frame.
- If the Gauss–Bonnet term is presented, then the function $\xi(\phi)$ plays a role of the effective potential, fully determining the position and stability of the de Sitter solutions, because

$$V_{eff} = \frac{2}{3}F - \frac{1}{C}. \tag{24}$$

- So, values of ϕ_{dS} satisfy the condition $F'(\phi_{dS}) = 0$. From Eq. (18), it follows

$$\lambda_{\pm} = -\frac{\sqrt{6CU_{dS}}}{4} \pm \frac{\sqrt{6CU_{dS}[9(3{U'_{dS}}^2 + U_{dS}) - 16CU_{dS}^2F''_{dS}]}}{12\sqrt{3{U'_{dS}}^2 + U_{dS}}}. \tag{25}$$

- For $U_{dS} > 0$, a de Sitter solution is unstable at $F''_{dS} < 0$ and stable at $F''_{dS} > 0$.

Note that the only difference between minimal and non-minimal coupling cases is that values of the Hubble parameter at de Sitter points $H_{dS}^2 = \frac{C}{6}U(\phi_{dS})$, can be different if U is not a constant.

CONCLUSIONS

• We analyze the Einstein–Gauss–Bonnet gravity model:

$$S=\int d^4x\sqrt{-g}\left(U(\phi)R-\frac{c}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-V(\phi)-F(\phi)\mathcal{G}\right),$$

• We have shown that, in the case of $U(\phi)>0$, it is possible to introduce the effective potential V_{eff} which can be expressed through the coupling function U, the scalar field potential V and the coupling function with the Gauss–Bonnet term F:

$$V_{eff}=\frac{2}{3}F-\frac{U^2}{V}.$$

• For $c\geqslant 0$, it is convenient to investigate the structure of fixed points using the effective potential, indeed, the stable de Sitter solutions correspond to minima of the effective potential V_{eff}.
• The effective potential V_{eff} can be used to analyze the stability of de Sitter solutions in model with $W(\mathcal{G})$ term that correspond to the case of $c=0$.

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• The effective potential V_{eff} can be used to analyze the stability of de Sitter solutions in model with $W(\mathcal{G})$ term that correspond to the case of $c = 0$.

Thank for your attention