Proceedings

Coupling to matter in degenerate scalar-tensor theories

Sebastian Garcia-Saenz [1,*] and Cédric Deffayet [2,3]

[1] Theoretical Physics, Blackett Laboratory, Imperial College London, SW7 2AZ London, U.K.
[2] Sorbonne Université, UPMC Paris 6 and CNRS, UMR 7095, Institut d'Astrophysique de Paris, GReCO, 98bis boulevard Arago, 75014 Paris, France
[3] IHES, Le Bois-Marie, 35 route de Chartres, 91440 Bures-sur-Yvette, France
* Correspondence: s.garcia-saenz@imperial.ac.uk

Version February 1, 2021 submitted to Universe

Abstract: Scalar-tensor theories of gravity provide an intriguing and compelling approach to the dark energy problem. They have received increased attention in recent years thanks to a wealth of developments both in the theoretical and experimental sides. The class of models known as "degenerate" provide a particularly interesting proposal. These theories extend general relativity by a single degree of freedom, despite their equations of motion being higher than second order, a virtue made possible by the existence of an additional constraint that removes the would-be instability associated to a ghost. This note presents a brief overview of the problem of matter coupling in degenerate scalar-tensor theories. It has been remarked that the presence of matter fields minimally coupled to the metric tensor can obstruct the degeneracy constraint, thus impairing the consistency of the theory. We explain through some illustrative examples the precise ways in which the extra degree of freedom may reappear. This occurs in the Hamiltonian language through a loss of constraints, which may happen either when the kinetic matrix is not block-diagonal in the presence of matter fields, or when the matter sector itself has constraints. We next turn to the more physically relevant case of fermionic matter, and show that spin-1/2 fermions evade these issues and can thus be consistently coupled to degenerate theories of scalar-tensor gravity.

Keywords: Dark Energy; Alternative Theories of Gravity; Scalar-Tensor Theories

1. Introduction

The extension of general relativity (GR) by additional light degrees of freedom is arguably the most natural way to provide a dynamical explanation of Dark Energy, thereby dispensing of the cosmological constant as the source of the observed late-time cosmic acceleration. Considering a single scalar field in addition to the metric tensor is, in this regard, particularly well motivated. These so-called scalar-tensor theories of gravity thus provide the most minimal modification of Einstein gravity in terms of local degrees of freedom and under some standard assumptions such as Poincaré invariance and locality. This is a virtue both from the theoretical and experimental perspectives, as its relative simplicity allows for strong analytical control while maintaining much of the phenomenology of GR. It is also not the least telling case for scalar-tensor theories that a related mechanism was likely to be at work during the pre-Big Bang epoch.

The complete classification of scalar-tensor theories thus seems to be an interesting and timely theoretical problem. In this effort, the assumption of having precisely three local degrees of freedom — two propagated by the metric and one by the scalar field — severely restricts the space of possible models. Although the physically meaningful question should make a distinction of light versus heavy degrees of freedom, it has nevertheless proved fruitful to demand the strict absence of additional fields beyond the aforementioned three, seeing that the resulting models often enjoy interesting properties that may have been difficult to discover through a more agnostic construction based on the rules of effective field theory.

This restriction on the number of degrees of freedom makes the classification problem mathematically well defined, although not easy as it turns out. Given the symmetries of the theory, it is sufficient to demand second order field equations, and taking this as a premise the problem has indeed been fully solved. The solution is given by Horndeski's scalar-tensor theory [1]. The remarkable observation is that this premise is however not a necessary one. That is, higher order equations of motion are not necessarily associated to extra unwanted degrees of freedom — unwanted indeed as they are generically associated to ghost-type instabilities according to the Ostrogradski theorem. This is so because the equations may happen to be degenerate, in the sense that a subset of them follows as a consequence of the others, implying in particular a reduction of the number of pieces of initial data that one would have naively inferred. The development and classification of these so-called degenerate scalar-tensor theories has been active research program in the past decade [2–6]. New models have been discovered throughout the years and have been given different names. We will refer to all of them collectively as DHOST, an acronym that stands for "Degenerate Higher-Order Scalar-Tensor" theories.

DHOST theories provide then a very interesting solution to the classification problem of scalar-tensor gravity. They are consistent theories within the scope of that problem, at least according to the way we have formulated it, although it is clear that physical consistency will reduce the space of allowed models by the imposition of further constraints. Most of these constraints arise from experimental tests of gravity, although here we will not be concerned with them — not because they are not important, but because their importance is crucially contingent on the physical context. For instance constraints derived from cosmological experiments need not apply on the scales of compact astrophysical objects. Theoretical constraints on the other hand have the chance to be more generally applicable, even if experiments must have the last word.

One such theoretical constraint that has remained largely overlooked is the question on the consistency of matter coupling in DHOST theories. The fact that matter fields can be problematic is seen easily in the Hamiltonian language, in which the degeneracy of the field equations is manifested in the form of a constraint on the phase space variables. The mixing with matter fields can then obstruct this constraint, leading to the reappearance of the ghost degree of freedom and an inconsistent theory. This may occur even if matter is minimally coupled to the metric tensor, for an indirect coupling with the DHOST scalar is still present. It is worth remarking that this issue is of course not specific to DHOST theories and may happen whenever two theories, where either or both have constraints when considered separately, are coupled in some way. It is thus a virtue of the Hamiltonian language to make it manifest that the degeneracy condition is in true a constraint, on equal footing to other constraints.

Understanding the precise ways in which the DHOST constraint may be lost was the subject of our work [7]. An additional result, and the most physically relevant in our view, is that spin-1/2 fermions can be coupled consistently with DHOST gravity, thus lending further support to the robustness of the theory. In the remaining of this note we present a summary of the main results, referring the interested reader to [7] for details and relevant literature.

2. Pathological matter fields in DHOST theory

As explained in the introduction, the potential pathologies associated to matter fields are manifested in a loss of constraints in the context of the Hamiltonian formalism. The first step is then to perform a 3+1 decomposition of the DHOST Lagrangian in terms of ADM variables [8]. We focus our attention to the subset of DHOST theories that are at most quadratic in the second derivative of the scalar field,

$$S_g[g,\phi] = \int d^4x\sqrt{-g}\Big[F(\phi,X)R + P(\phi,X) + Q(\phi,X)\Box\phi + C^{\mu\nu\rho\sigma}[\phi]\nabla_\mu\nabla_\nu\phi\nabla_\rho\nabla_\sigma\phi\Big]\,. \tag{1}$$

Here $X := \nabla^\mu\phi\nabla_\mu\phi$, R is the 4-dimensional curvature scalar, and

$$\begin{aligned} C^{\mu\nu\rho\sigma} &:= A_1 g^{\mu(\rho}g^{\sigma)\nu} + A_2 g^{\mu\nu}g^{\rho\sigma} + \frac{A_3}{2}\left(\phi^\mu\phi^\nu g^{\rho\sigma} + \phi^\rho\phi^\sigma g^{\mu\nu}\right) \\ &+ \frac{A_4}{2}\left(\phi^\mu\phi^{(\rho}g^{\sigma)\nu} + \phi^\nu\phi^{(\rho}g^{\sigma)\mu}\right) + A_5\phi^\mu\phi^\nu\phi^\rho\phi^\sigma\,, \end{aligned} \tag{2}$$

where $\phi_\mu := \nabla_\mu\phi$ and the A's are functions of ϕ and X. It is then a straightforward calculation to derive the action in 3+1 form. The result is

$$\begin{aligned} S_g = \int dt d^3x \Big\{ & N\sqrt{\gamma}\Big[\mathcal{A}V_*^2 + 2\mathcal{B}^{ij}V_*K_{ij} + \mathcal{K}^{ij,kl}K_{ij}K_{kl} + 2\mathcal{C}^{ij}K_{ij} + 2\mathcal{C}^0V_* - \mathcal{U}\Big] \\ & + \lambda^0\left(NA_* + N^iA_i - \dot{\phi}\right) + \lambda^i\left(A_i - D_i\phi\right)\Big\}\,, \end{aligned} \tag{3}$$

where N and N^i are the lapse and shift ADM variables, K_{ij} is the extrinsic curvature, and D_i is the covariant derivative compatible with the 3-metric γ_{ij}. The auxiliary vector field A_μ is constrained to be equal to $\nabla_\mu\phi$ by the Lagrange multiplier λ^μ, and is necessary to produce a Lagrangian with only first time derivatives from which the canonical momenta can be defined unambiguously. The above 3+1 action also includes the definitions

$$A_* := n^\mu A_\mu = \frac{1}{N}(A_0 - N^iA_i)\,, \qquad V_* := \frac{1}{N}\left(\dot{A}_* - A^iD_iN - N^iD_iA_*\right)\,, \tag{4}$$

needed to eliminate non-linear terms in the lapse and shift from the Hamiltonian []. Finally the tensors $\mathcal{A}$, $\mathcal{B}^{ij}$, $\mathcal{K}^{ij,kl}$, $\mathcal{C}^{ij}$, $\mathcal{C}^0$ and $\mathcal{U}$ are constructed from the functions defining the theory in (1), and depend only on the variables ϕ, A_*, A_i and γ_{ij}, but not their time derivatives or the lapse and shift. Explicit expressions are given in [7]. It suffices here to note that a DHOST theory of the class we are discussing is characterized by the identity

$$\mathcal{A} - \mathcal{K}^{-1}_{ij,kl}\mathcal{B}^{ij}\mathcal{B}^{kl} = 0\,. \tag{5}$$

This is the condition that ensures the degeneracy of the field equations, or equivalently the presence of an additional constraint in the Hamiltonian formalism, within pure DHOST gravity. The explicit form of the DHOST constraint in the absence of matter is given by

$$\Psi := p_* - 2\mathcal{K}^{-1}_{ij,kl}\pi^{ij}\mathcal{B}^{kl} + 2\sqrt{\gamma}\left(\mathcal{K}^{-1}_{ij,kl}\mathcal{C}^{ij}\mathcal{B}^{kl} - \mathcal{C}^0\right) \approx 0\,, \tag{6}$$

where p_* and π^{ij} are the canonical momenta conjugate to A_* and γ_{ij}, respectively.

We are now in position to understand more precisely the ways in which the degeneracy of DHOST gravity may be lost in the presence of matter fields. We can classify the pathological matter theories into two types:

(I) The constraint Ψ is lost, and no analogue of it exists.

This will be the case when the rank of the Hessian matrix

$$\mathcal{H}_{IJ} := \frac{\partial^2\mathcal{L}}{\partial\dot{\psi}^I\partial\dot{\psi}^J}\,, \tag{7}$$

(here ψ^I stands for all the fields) is greater than the sum of the ranks of the DHOST and matter Hessians that one would have in the absence of coupling. This cannot occur when the full Hessian is block-diagonal in the DHOST and matter variables. As we are restricting our attention to minimal matter coupling, any matter Lagrangian that does not involve the Christoffel connection will lead to a block-diagonal Hessian and thus be safe according to this criterion. The converse of this is of course not true. Although a non-block-diagonal Hessian is at risk of failing this consistency check, it may still enjoy a (possibly modified) degeneracy constraint.

(II) The constraint Ψ (or some analogue of it) does exist, but it fails to Poisson-commute with one or more constraints present in the matter sector.

In the absence of matter the DHOST constraint Ψ is a primary, second-class constraint, and it Poisson-commutes with all the other primary constraints in the gravity sector. It therefore leads to a secondary constraint, which together with Ψ is responsible for removing the would-be ghost degree of freedom. If now the matter sector itself has some constraints, there is the risk that they may not commute with Ψ, implying the loss of the associated secondary constraint and the reappearance of the unwanted degree of freedom.

It is not difficult to find examples that fail either of these two criteria. A matter field that fails criterion (I) is provided by a vector with a non-Maxwell kinetic structure,

$$S_m = \int d^4x\sqrt{-g}\,\nabla^\mu B^\nu \nabla_\mu B_\nu \,. \tag{8}$$

This action involves the Christoffel connection and hence the time derivative of the 3-metric in the ADM language. The full Hessian matrix is therefore not block-diagonal and it is not hard to show, for instance through an explicit computation of the rank, that the DHOST constraint is lost [7]. Although this matter theory is pathological in itself, already without gravity, it is consistent as far as the above criteria are concerned in the context of pure GR. Thus its failure to satisfy the criteria within DHOST shows that an extra ghost must be present, irrespective of whether the matter degrees of freedom are themselves healthy or not.

An example of a matter theory that fails criterion (II) is given by a cubic galileon (as noted first in [9])

$$S_m = \int d^4x\sqrt{-g}\left[-\frac{1}{2}(\nabla\pi)^2 + \kappa(\nabla\pi)^2\Box\pi\right], \tag{9}$$

where κ is a coupling constant. When minimally coupled to DHOST one finds that the full Hessian matrix is not block-diagonal, yet a modified degeneracy constraint still exists. Now however the matter action has itself a constraint (associated to a degeneracy of the 3+1 formulation of the galileon Lagrangian) which fails to Poisson-commute with the DHOST constraint [7]. The coupled theory thus propagates more degrees of freedom than the uncoupled gravity and matter sectors and it must therefore be deemed pathological.

3. Spinor fields in DHOST theory

The coupling of spinor fields to theories of modified gravity is a question of obvious theoretical importance. Our findings show that inspecting this issue in the context of DHOST gravity is all the more relevant in view of the possibility that the above criteria for consistent coupling might a priori be violated by spinor fields. Indeed fermionic Lagrangians are in some sense doubly dangerous as they couple to the spin connection, leading therefore to a non-block-diagonal kinetic Hessian, and are also subject to constraints of their own. Thus they risk violating both criteria (I) and (II).

The positive result derived in our work [7] is that spinor fields are in fact consistent, at least for a fairly broad class of models. We summarize our analysis for the case of a linear Majorana spin-1/2 fermion, although our results apply more generally as we comment on the final section. We consider then the Majorana action minimally coupled to gravity,

$$S_m = -\frac{1}{2}\int d^4x\sqrt{-g}\,e_a{}^\mu \lambda^\alpha (\gamma^a)_\alpha{}^\beta \nabla_\mu \lambda_\beta \,, \tag{10}$$

where ${e_a}^\mu$ is the (inverse) tetrad field, while the covariant derivative of λ_α (we use 4-component notation for spinors),

$$\nabla_\mu \lambda_\alpha = \partial_\mu \lambda_\alpha + \frac{1}{4}\, {\omega^{ab}}_\mu {(\gamma_{ab})_\alpha}^\beta \lambda_\beta \,, \tag{11}$$

depends on the spin connection ${\omega^{ab}}_\mu$ as announced. As we work in the second-order formalism, we have ${\omega^a}_{b\mu} = {e_b}^\nu (\Gamma^\rho_{\mu\nu} {e^a}_\rho - \partial_\mu {e^a}_\nu)$ and hence a mixture of time derivatives of the spinor and tetrad variables.

In spite of the mixing of velocities, it turns out that the DHOST constraint is unaffected. This result is non-trivial given that the tetrad canonical momentum,

$${\pi_a}^i = 2\sqrt{\gamma}\, e_{aj} \left[\mathcal{K}^{ij,kl} K_{kl} + \mathcal{B}^{ij} V_* + \mathcal{C}^{ij} \right] + \frac{1}{8}\sqrt{\gamma}\, \lambda^\alpha {(\gamma_a E^i E^0)_\alpha}^\beta \lambda_\beta \,, \tag{12}$$

is explicitly modified by the terms containing the spinor variables. However it remains true that the combination

$$\pi^{ij} := \frac{1}{2}\, e^{a(i} {\pi_a}^{j)} = \sqrt{\gamma} \left[\mathcal{K}^{ij,kl} K_{kl} + \mathcal{B}^{ij} V_* + \mathcal{C}^{ij} \right] \,, \tag{13}$$

is independent of the spinor field, and it is precisely this expression, rather than ${\pi_a}^i$ itself, which defines the DHOST constraint. We have further found that this is not an accident of the linearity of the matter theory but actually holds for any Majorana spin-1/2 action that is linear in $\nabla_\mu \lambda^\alpha$ but completely general in non-derivative self-interactions.

The final check is that the DHOST constraint still Poisson-commutes with all other primary constraints and so generates the necessary secondary constraint. The fact that the Majorana Lagrangian is linear in $\nabla_\mu \lambda^\alpha$ means that the spinor canonical momentum is subject to a constraint. It is then easy to verify that this constraint indeed commutes with the constraint Ψ of the gravity sector and so the second criterion is also satisfied.

4. Final remarks

Most phenomenological applications of DHOST gravity have only dealt with standard matter fields for which the pathologies we have uncovered do not apply. Spin-0 matter particles described (in flat space) by Lagrangians of the form $\mathcal{L} = P(X, \pi)$, with $X = -\frac{1}{2}(\partial\pi)^2$, are immediately safe since minimal coupling to gravity does not introduce mixings with the connection and hence maintains the block-diagonal structure of the Hessian matrix. Note that perfect fluids are contained within this class. The same conclusion applies to standard spin-1 theories, i.e. Maxwell, Proca and (massive) Yang–Mills theories. Yet we see no convincing reason why matter fields in the context of DHOST should be restricted to these simple cases, at least as a matter of principle. More generally, we expect our results to be important for any model of modified gravity characterized by a degeneracy condition.

Our findings concerning spinor fields are positive and encouraging, but not fully conclusive. We focused on a matter sector containing a single Majorana spin-1/2 field. If one restricts the attention to linear theories (i.e. theories that are free in the absence of gravity) then all our conclusions remain unchanged. In particular a linear Dirac spinor, being essentially the sum of two Majorana fields, is consistent according to our criterion. Mutually interacting spinor fields are however not covered by our results and it would be interesting to address this type of models. It would also be intriguing to see if a spin-3/2 field can be coupled consistently to DHOST gravity as this may be of relevance to the problem of supersymmetrizing generalized scalar-tensor theories.

Funding: This research was funded by the European Research Council under the European Community's Seventh Framework Programme (FP7/2007-2013 Grant Agreement no. 307934, NIRG project) and by the European Union's Horizon 2020 Research Council grant 724659 MassiveCosmo ERC-2016-COG.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Horndeski, G.W. Second-order scalar-tensor field equations in a four-dimensional space. *Int. J. Theor. Phys.* **1974**, *10*, 363–384. doi:10.1007/BF01807638.
2. Zumalacárregui, M.; García-Bellido, J. Transforming gravity: from derivative couplings to matter to second-order scalar-tensor theories beyond the Horndeski Lagrangian. *Phys. Rev.* **2014**, *D89*, 064046, [arXiv:gr-qc/1308.4685]. doi:10.1103/PhysRevD.89.064046.
3. Gleyzes, J.; Langlois, D.; Piazza, F.; Vernizzi, F. Healthy theories beyond Horndeski. *Phys. Rev. Lett.* **2015**, *114*, 211101, [arXiv:hep-th/1404.6495]. doi:10.1103/PhysRevLett.114.211101.
4. Langlois, D.; Noui, K. Degenerate higher derivative theories beyond Horndeski: evading the Ostrogradski instability. *JCAP* **2016**, *1602*, 034, [arXiv:gr-qc/1510.06930]. doi:10.1088/1475-7516/2016/02/034.
5. Crisostomi, M.; Koyama, K.; Tasinato, G. Extended Scalar-Tensor Theories of Gravity. *JCAP* **2016**, *1604*, 044, [arXiv:hep-th/1602.03119]. doi:10.1088/1475-7516/2016/04/044.
6. Ben Achour, J.; Crisostomi, M.; Koyama, K.; Langlois, D.; Noui, K.; Tasinato, G. Degenerate higher order scalar-tensor theories beyond Horndeski up to cubic order. *JHEP* **2016**, *12*, 100, [arXiv:hep-th/1608.08135]. doi:10.1007/JHEP12(2016)100.
7. Deffayet, C.; Garcia-Saenz, S. Degeneracy, matter coupling, and disformal transformations in scalar-tensor theories. *Phys. Rev. D* **2020**, *102*, 064037, [arXiv:hep-th/2004.11619]. doi:10.1103/PhysRevD.102.064037.
8. Langlois, D.; Noui, K. Hamiltonian analysis of higher derivative scalar-tensor theories. *JCAP* **2016**, *1607*, 016, [arXiv:gr-qc/1512.06820]. doi:10.1088/1475-7516/2016/07/016.
9. de Rham, C.; Matas, A. Ostrogradsky in Theories with Multiple Fields. *JCAP* **2016**, *1606*, 041, [arXiv:hep-th/1604.08638]. doi:10.1088/1475-7516/2016/06/041.

© 2021 by the authors. Submitted to *Universe* for possible open access publication under the terms and conditions of the Creative Commons Attribution (CC BY) license (http://creativecommons.org/licenses/by/4.0/).