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Role of anisotropy on the tidal deformability of compact stellar objects

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Abstract: In this paper, we introduce a framework to study the tidal deformation of relativistic anisotropic compact stars. Anisotropic stresses are ubiquitous in nature and widely used in modelling compact stellar object. Tidal deformability of astrophysical compact objects is a natural effect of gravity such as one produced by a companion in a binary system. In general relativity, the existence of this measurable effect of gravity can be quantified by their tidal Love numbers (TLN) which characterize the deformability of a neutron star (NS) from sphericity. The tidal deformability or polarizability parameter of a NS depends on its complex internal structure and hence the nature of the compact object can study through measuring the TLN. We choose a particular solution which is the anisotropic generalization of Tolman IV model as the interior of the compact stellar object. The physical acceptability of the model has been shown graphically by considering the pulsar $4U$ 1608-52 with their current estimated mass and radius. By computing quadrupole moment we found out the TLN as a dependent on anisotropy of the compact object. We graphically analyze the variation of the TLN against anisotropy for different compact objects with compactness factor. The numerical value of TLN is given for different compact objects for physically acceptable value of anisotropic parameter.

Keywords: Compact star; Anisotropy; Tidal effect; Love number

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1. Introduction

Compact objects are the extremely dense astro-physical objects provides strong gravity and high density for studying the fundamental physics related to nuclear matter properties. In general, compact objects exist with its binary companion is a natural setup. In this binary setup a compact star is assumed to be immersed in the tidal field of its companion hence produce tidal deformation. The possibility of generation of the gravitational waves during extremely fast rotational motion of binary system was predicted. Recently gravitational wave has been detected by advanced astronomical observations of LIGO and Virgo collaborations from the binary neutron star merger event GW170817 [1]. A neutron star placed in a perturbing external gravitational field is deformed and induces quadrupole moment which affect the binding energy of the system and increase the rate of emission of gravitational waves during the late stage inspirals. The recent observational data based on the measurement of the tidal deformability impose a stringent constraint on the allowed equation of state (EOS). In

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particular, the theoretically prediction of the mass and radius of neutron star (NS) mostly depends on the nature of the nuclear EOS at supra-nuclear densities. In this context tidal deformability can be used to study their interiors. The EOS of neutron stars, involves in their microscopic properties and uniquely determines the macroscopic properties, such as the maximum allowed NS mass, radius and tidal effects. The tidal response is the astrophysical constraints that can be employed as probes of NS properties. It is the astrophysically observable macroscopic property of NS which can be defined as the ratio of the induced multipole moment of a star over the inducing tidal field from its companion. The tidal Love number (TLN), which is the ratio of the induced quadrupole moment to the perturbing tidal gravitational field, can be expressed by a relatively simple analytical formula.

On the other hand the existence of pressure anisotropy, the difference of radial and transverse pressures, is ubiquitous in a compact star. The source of pressure anisotropy in a compact star can be due to various reasons, e.g., pion and kaon condensates [2,3], high density, existence of a solid core or type $3A$ superfluid [4,5], strong magnetic fields [6], a mixture of perfect and a null fluid, viscosity, phase transition [7] etc. There are several works available in the literature where incorporating anisotropy into the matter distribution of compact objects in the background of General Relativity (GR), have been addressed to various issues of the compact structures [8–14].

In this paper, we assume a known solution which is an anisotropic generalization of Tolman IV model to describe the compact star. We have calculated the TLN which actually measures the tidal deformability of the compact object induced by external field.

2. Physical features and tidal Love number

The tidal distortion of NSs in a binary system connects the EOS, describing the nature of the matter composition star with that of the gravitational wave emission during the inspiral [15]. We consider a static spherically symmetric star, immersed in an external quadrapolar tidal field $\mathcal{E}_{ij}$ [16,17] arising due to its binary companion. The star in response to tidal field develops a quadrapole moments $\mathcal{Q}_{ij}$ which can be related to the linear order external tidal field $\mathcal{E}_{ij}$ as [17]

$$\mathcal{Q}_{ij} = -\Lambda\, \mathcal{E}_{ij}, \tag{1}$$

where Λ is the tidal deformability of the NS and is related to the $l = 2$ dimensionless TLN k_2 as [17]

$$k_2 = \frac{3}{2}\Lambda\, R^{-5}. \tag{2}$$

The background geometry of spacetime of a spherical static star can be written as

$$\begin{aligned} {}^{(0)}ds^2 &= {}^{(0)}g_{\mu\nu}dx^\mu dx^\nu \\ &= -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \end{aligned} \tag{3}$$

For the spherically static metric (3), the stress-energy tensor is given as

$$^{(0)}T^{\xi}_{\chi} = (\rho + p_t)u^{\xi}u_{\chi} + p_t g^{\xi}_{\chi} + (p_r - p_t)\eta^{\xi}\eta_{\chi}, \tag{4}$$

where $u^{\xi}u_{\xi} = -1$, $\eta^{\xi}\eta_{\xi} = 1$ and $\eta^{\xi}u_{\xi} = 0$.

We choose a particular model which is an anisotropic generalization of Tolman IV Model [18] given as

$$e^{2\nu} = A^2(1 + aCr^2), \tag{5}$$

$$e^{2\lambda} = \frac{1}{\frac{(aCr^2+1)(1-BCr^2)}{2aCr^2+1} - \frac{\alpha Cr^2}{2aCr^2+1}}. \tag{6}$$

Interesting to note that for $\alpha = 0$, this solution reduces to the well known Tolman IV solution [19].

For the line element (3), the independent set of the Einstein field equations are then obtained as

$$
\begin{aligned}
8\pi\rho &= \frac{1}{r^2}\left[r(1-e^{-2\lambda})\right]', && (7)\\
8\pi p_r &= -\frac{1}{r^2}\left(1-e^{-2\lambda}\right) + \frac{2\nu'}{r}e^{-2\lambda}, && (8)\\
8\pi p_t &= e^{-2\lambda}\left(\nu'' + \nu'^2 + \frac{\nu'}{r} - \nu'\lambda' - \frac{\lambda'}{r}\right), && (9)
\end{aligned}
$$

where primes ($'$) denote differentiation with respect to r. In the field Equations (7)-(9), we have assumed $G = 1 = c$. The system of equations determines the behavior of the gravitational field of an anisotropic imperfect fluid sphere.

For the assumed solutions we have

$$
\begin{aligned}
8\pi\rho &= \frac{C\left(a\left(Cr^2\left(a\left(6BCr^2+2\right)+2\alpha+7B\right)+3\right)+3(\alpha+B)\right)}{\left(2aCr^2+1\right)^2}, && (10)\\
8\pi p_r &= -\frac{C\left(aCr^2+1\right)\left(a\left(3BCr^2-1\right)+B\right)+\alpha C\left(3aCr^2+1\right)}{(aCr^2+1)(2aCr^2+1)}, && (11)\\
8\pi p_t &= -\frac{C\left(aCr^2+1\right)^2\left(a\left(3BCr^2-1\right)+B\right)+\alpha C\left(aCr^2\left(aCr^2+3\right)+1\right)}{\left(aCr^2+1\right)^2(2aCr^2+1)}, && (12)\\
8\pi\Delta &= \frac{a\alpha C^2 r^2}{\left(aCr^2+1\right)^2}, && (13)
\end{aligned}
$$

where we define $\Delta = 8\pi(p_t - p_r)$ as the measure of anisotropy of the spherical system.

The exterior Schwarzschild metric

$$
ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \qquad (14)
$$

across the boundary boundary of the star $r = R$, where M is the total mass of the sphere.

Making use of the junction conditions, the constants A, B, C are determined as

$$
\begin{aligned}
A &= \frac{\sqrt{R-3M}}{\sqrt{R}}, && (15)\\
C &= \frac{M}{aR^3 - 3aMR^2}, && (16)\\
B &= \frac{(R-3M)(a(R-2M)-\alpha R)}{R(R-2M)}. && (17)
\end{aligned}
$$

Now the background metric ${}^{(0)}g_{\mu\nu}(x^\nu)$ of a NS in effect to external tidal field with small perturbation $h_{\mu\nu}(x^\nu)$ gets modified as

$$
g_{\mu\nu}(x^\nu) =^{(0)} g_{\mu\nu}(x^\nu) + h_{\mu\nu}(x^\nu). \qquad (18)
$$

For the linearized metric perturbation $h_{\mu\nu}$, using the method as in [20,21], without loss of generality, we restrict ourselves to static $l = 2$, $m = 0$ even parity perturbation. The perturbed metric with the assumption that the tidal deformation will be axis symmetric around the line connecting the two stars which we take as the axis of spherical harmonic decomposition becomes as

$$
h_{\mu\nu} = \mathrm{diag}\left[H_0(r)e^{2\nu}, H_2(r)e^{2\lambda}, r^2K(r), r^2\sin^2\theta K(r)\right]Y_{2m}(\theta,\phi). \qquad (19)
$$

Furthermore the perturbed energy momentum tensor is defined by $T^{\xi}_{\chi} =^{(0)} T^{\xi}_{\chi} + \delta T^{\xi}_{\chi}$, where the non-zero components of T^{ξ}_{χ} are: $\delta T^{t}_{t} = -\dfrac{d\rho}{dp_r}\,\delta p_r\, Y(\theta,\phi)$, $\delta T^{r}_{r} = \delta p_r(r)Y(\theta,\phi)$, & $\delta T^{\theta}_{\theta} = \delta T^{\phi}_{\phi} = \frac{dp_t}{dp_r}\delta p_r(r)Y(\theta,\phi)$. With these perturbed quantities we can write down the perturbed Einstein field equation as follows:

$$G^{\xi}_{\chi} = 8\pi T^{\xi}_{\chi}, \tag{20}$$

68 where the Einstein tensor G^{ξ}_{χ} is calculated using the metric $g_{\chi\xi}$.

From different components of the background Einstein field equation ${}^{(0)}G^{\xi}_{\chi} = 8\pi^{(0)}T^{\xi}_{\chi}$, we can have the following relationships:

$$ {}^{(0)}G^{t}_{t} = 8\pi^{(0)}T^{t}_{t} \Rightarrow \lambda'(r) = \frac{8\pi r^2 e^{2\lambda(r)}\rho(r) - e^{2\lambda(r)} + 1}{2r}, \tag{21}$$

$$ {}^{(0)}G^{r}_{r} = 8\pi^{(0)}T^{r}_{r} \Rightarrow \nu'(r) = \frac{8\pi r^2 p_r(r) e^{2\lambda(r)} + e^{2\lambda(r)} - 1}{2r}, \tag{22}$$

69 where and hereafter the prime denotes the derivative w.r.t. the radial coordinate r.

Also we know that $\nabla^{(0)}_{\xi} T^{\xi}_{\chi} = 0$. Now choosing $\xi = r$, by expanding and solving the equation, we can find the expression

$$p_r'(r) = \frac{-rp_r(r)\nu'(r) - 2p_r(r) + 2p_t(r) - r\rho(r)\nu'(r)}{r}. \tag{23}$$

Again from the various components of the perturbed Einstein Equation (20), we get the following relations

$$G^{\theta}_{\theta} - G^{\phi}_{\phi} = 0 \Rightarrow H_0(r) = H_2(r) = H(r), \tag{24}$$

$$G^{\theta}_{r} = 0 \Rightarrow K' = H' + 2H\nu', \tag{25}$$

$$G^{\theta}_{\theta} + G^{\phi}_{\phi} = 8\pi(T^{\theta}_{\theta} + T^{\phi}_{\phi}) \Rightarrow \delta p_r = \frac{H(r)e^{-2\lambda(r)}(\lambda'(r) + \nu'(r))}{8\pi \frac{dp_t}{dp_r} r}. \tag{26}$$

Using the identity

$$\frac{\partial^2 Y(\theta,\phi)}{\partial\theta^2} + cot(\theta)\frac{\partial Y(\theta,\phi)}{\partial\theta} + \csc^2(\theta)\frac{\partial^2 Y(\theta,\phi)}{\partial\phi^2} = -6Y(\theta,\phi)$$

and Equations (21) – (26), we have the master equation for $H(r)$ as

$$-\frac{1}{e^{-2\lambda(r)}Y(\theta,\phi)}[G^{t}_{t} - G^{r}_{r}] = -\frac{8\pi}{e^{-2\lambda(r)}Y(\theta,\phi)}[T^{t}_{t} - T^{r}_{r}]$$
$$\Rightarrow H''(r) + \mathcal{R}H'(r) + \mathcal{S}H(r) = 0, \tag{27}$$

where

$$\mathcal{R} = -\left(\frac{-e^{2\lambda(r)} - 1}{r} - 4\pi r e^{2\lambda(r)}(p_r(r) - \rho(r))\right), \tag{28}$$

$$\mathcal{S} = -\left(\frac{4e^{2\lambda(r)} + e^{4\lambda(r)} + 1}{r^2} + 64\pi^2 r^2 p_r(r)^2 e^{4\lambda(r)} + 16\pi e^{2\lambda(r)}\left(p_r(r)\left(e^{2\lambda(r)} - 2\right)\right.\right.$$
$$\left.\left. - p_t(r) - \rho(r)\right) + \frac{-4\pi \frac{d\rho}{dp_r} e^{2\lambda(r)}(p_r(r) + \rho(r)) - 4\pi e^{2\lambda(r)}(p_r(r) + \rho(r))}{\frac{dp_t}{dp_r}}\right). \tag{29}$$

The vacuum exterior of the star is of Schwarzschild type, so that by setting $\rho = 0$, $p_r = 0$, $p_t = 0$ and $e^{2\lambda} = 1/(1-2M/r)$, the master Equation (27) becomes

$$-H''(r) - \frac{2(M-r)H'(r)}{r(2M-r)} + \frac{2H(r)(2M^2 - 6Mr + 3r^2)}{r^2(r-2M)^2} = 0. \tag{30}$$

The solution to this second order differential equation (30) is

$$H(r) = \frac{1}{2M^2 r(2M-r)}\left[c_2\left(-2M\left(2M^3 + 4M^2 r - 9Mr^2 + 3r^3\right) - 3r^2(r-2M)^2\right.\right.$$
$$\left.\left. \times \log\left(\frac{r}{M} - 2\right) + 3r^2(r-2M)^2 \log\left(\frac{r}{M}\right)\right)\right] + \frac{3c_1 r(2M-r)}{M^2}, \tag{31}$$

where c_1 and c_2 are integration constants. In order to obtain the expressions for these constants, lets do series expansion of Equation (31)

$$H(r) = -\frac{3c_1 r^2}{M^2} + \frac{6c_1 r}{M} - \frac{c_2(8M^3)}{5r^3} + \mathcal{O}\left(\left(\frac{1}{r}\right)^4\right). \tag{32}$$

Now at large r the metric coefficient g_{tt} is given by [17]:

$$\frac{(1-g_{tt})}{2} = -\frac{M}{r} - \frac{3\mathcal{Q}_{ij}}{2r^3}\left(n^i n^j - \frac{1}{3}\delta^{ij}\right) + \mathcal{O}\left(\frac{1}{r^4}\right) + \frac{1}{2}\mathcal{E}_{ij}x^i x^j + \mathcal{O}(r^3), \tag{33}$$

70 where $n^i = x^i/r$.

Matching the asymptotic solution from Equation (32) with the expansion from Equation (33) and using the Equation (1), we have

$$c_1 = -\frac{M^2\mathcal{E}}{3}, \quad c_2 = \frac{15\mathcal{Q}}{8M^3}. \tag{34}$$

Using Equations (34), (31) and (2), we obtain the expression for TLN k_2 as follows:

$$k_2 = [8(1-2\mathcal{C})^2\mathcal{C}^5(2\mathcal{C}(y-1) - y + 2)]/X, \tag{35}$$

where

$$X = (5(2\mathcal{C}(\mathcal{C}(2\mathcal{C}(\mathcal{C}(2\mathcal{C}(y+1) + 3y - 2) - 11y + 13) + 3(5y-8)) - 3y + 6)$$
$$+3(1-2\mathcal{C})^2(2\mathcal{C}(y-1) - y + 2)\log\left(\frac{1}{\mathcal{C}} - 2\right) - 3(1-2\mathcal{C})^2(2\mathcal{C}(y-1) - y + 2)\log\left(\frac{1}{\mathcal{C}}\right)\Big)\Big). \tag{36}$$

Here the Compactness $\mathcal{C} = \frac{M}{R}$ and y depends on r, H and its derivatives evaluated at R with

$$y = \left.\frac{rH'(r)}{H(r)}\right|_{r=R}. \tag{37}$$

To calculate numerically the value of k_2 for a particular NS [22], one need to modify the master Equation (27) using the Equation (37) as

$$ry' + y^2 + (r\mathcal{R} - 1)y + r^2\mathcal{S} = 0. \tag{38}$$

3. Results

The recent data available from the pulsar $4U\ 1608-52$ (for the star to be composed of an anisotropic fluid distribution with $\alpha = 0.5$) whose estimated mass and radius are $M = 1.57\ M_\odot$ and $R = 9.8\ km$, respectively [23,24] are used to find the constants are calculated as $A = 0.53953,\ B = 0.291097,\ C = 0.008452$. We set as $a = 1$. Making use of these values, we show graphically the nature of all the physically meaningful quantities in Fig. 1.

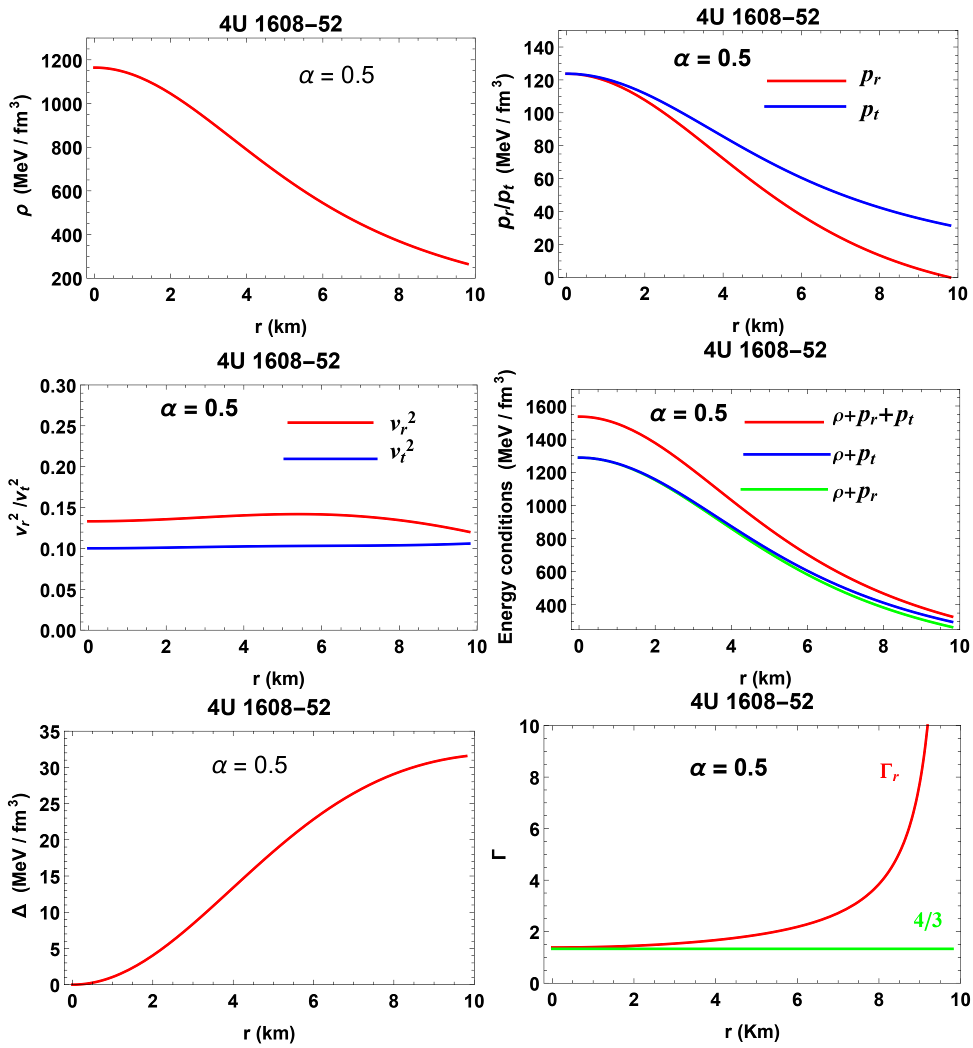

Figure 1. Physical features are plotted against the radial parameter for the compact star $4U\ 1608-52$.

Using the initial condition $y(0) = 2$ and all the mentioned equations, Equation (38) for a particular NS, can be numerically calculated. Having the numerical value of y, for a particular NS, from the Equation (35), the TLN k_2 can be obtained numerically.

4. Conclusions

The plots clearly show that all the quantities comply with the requirements of a physically viable realistic star. In particular, the figures highlight the effect of anisotropy on the gross physical behavior of the compact star. In Fig. 2, the TLN k_2 is plotted against

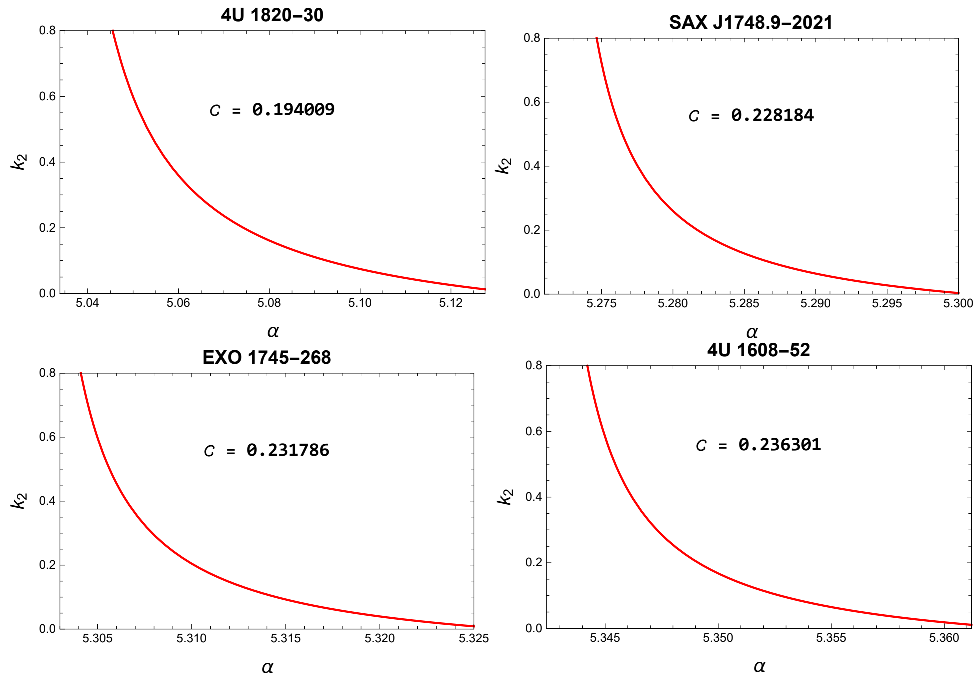

Figure 2. k_2 is plotted against α for different compact objects with compactness $\mathcal{C}$ only for the allowed values of α.

α for different compact objects with compactness $\mathcal{C}$. From this panel of figures we note that k_2 decreases monotonically with increasing α. This is a much expected physical property of a compact object with anisotropy.

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