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On Conditional Tsallis Entropy

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Tsallis entropy, a generalization of Shannon entropy, was extensively studied by Constantino Tsallis in 1988 and provides an alternative way of dealing with several characteristics of nonextensive physical systems given that the information about the intrinsic fluctuations in the physical system can be characterized by the nonextensivity parameter α. It can be applied to many scientific fields such as physics, economics, computer science, and biology.
It is known that as the parameter α approaches 1 the Tsallis entropy corresponds to the Shannon entropy.

$$T_\alpha(X) = \frac{1}{\alpha-1}\left(1-\sum_x \mathrm{P}(X=x)^\alpha\right), (\text{for } \alpha>0,\ \alpha\neq 1).$$

What is the definition of **conditional Tsallis entropy?**

There is no generally accepted definition for conditional Tsallis entropy.

Proposal definitions:

1. Definition of $T_\alpha(Y|X)$ from [1]:

$$T_\alpha(Y|X) = \sum_x P(X=x)^\alpha T_\alpha(Y|x) = \frac{1}{\alpha-1}\sum_x P(X=x)^\alpha\left(1-\sum_y P(Y=y|X=x)^\alpha\right)$$

One can easily verify that $T_\alpha(X,Y) = T_\alpha(Y|X) + T_\alpha(X)$ and therefore it satisfies the chain rule.

2. Definition of S(Y|X) from [2] (Definition 2.8):

$$S_\alpha(Y|X) = \sum_x P(X=x)T_\alpha(Y|X=x) = \sum_x P(X=x)\frac{1}{\alpha-1}\left(1-\sum_y P(Y=y|X=x)^\alpha\right) = \frac{1}{\alpha-1}\sum_x P(X=x)\left(1-\sum_y P(Y|y|X=x)^\alpha\right)$$

3. Definition of $S'_\alpha(Y|X)$ from [2] (Definition 2.10):

$$S'_\alpha(Y|X) = \frac{1}{\alpha-1}\left(\frac{1-\sum_{x,y}P(X=x,Y=y)^\alpha}{\sum_x P(X=x)^\alpha}\right)$$

4. Definition of $T'_\alpha(Y|X)$ (new proposal):

$$T'_\alpha(Y|X) = \frac{1}{\alpha-1}\,\text{máx}_x\left(1-\sum_y P(Y=y|X=x)^\alpha\right)$$

This new proposal is a natural approach to consider as possible definition. It defines the conditional value as the maximum value of all marginal distributions. Due to this fact, and similarly to what happens with Rényi's entropy (another generalization of Shannon entropy), this definition was also analyzed although it was never considered in the literature before.

Results

- The relationship between the four definitions, described in this work, are summarized in the Figure 1.
- In our understanding, it would be expectable that a proposal for conditional Tsallis entropy would satisfy the following properties: (1) Chain Rule; (2) Convergence to Shannon entropy as the parameter tended to 1; (3) Its value would be between 0 and the upper bound of the unconditional version. In Table 1, we summarize the properties that the four proposals have (we also added the property of being a non increasing function with α).

$f(Y\|X)$	$T_\alpha(Y\|X)$	$S_\alpha(Y\|X)$	$S'_\alpha(Y\|X)$	$T'_\alpha(Y\|X)$
Chain Rule	yes	no	no	no
$\lim_{\alpha\to1} f(Y\|X)$	$H(Y\|X)$	$H(Y\|X)$	$H(Y\|X)$	$\max_x H(Y\|X=x)$
$0\le f(Y\|X)\le \frac{\|Y\|^{1-\alpha}}{1-\alpha}$	$\alpha>1$	yes	$\alpha>1$	yes
f is non increasing with α	yes	yes	no	yes

Table 1: Summary of the properties of the proposal definitions of conditional Tsallis entropy.

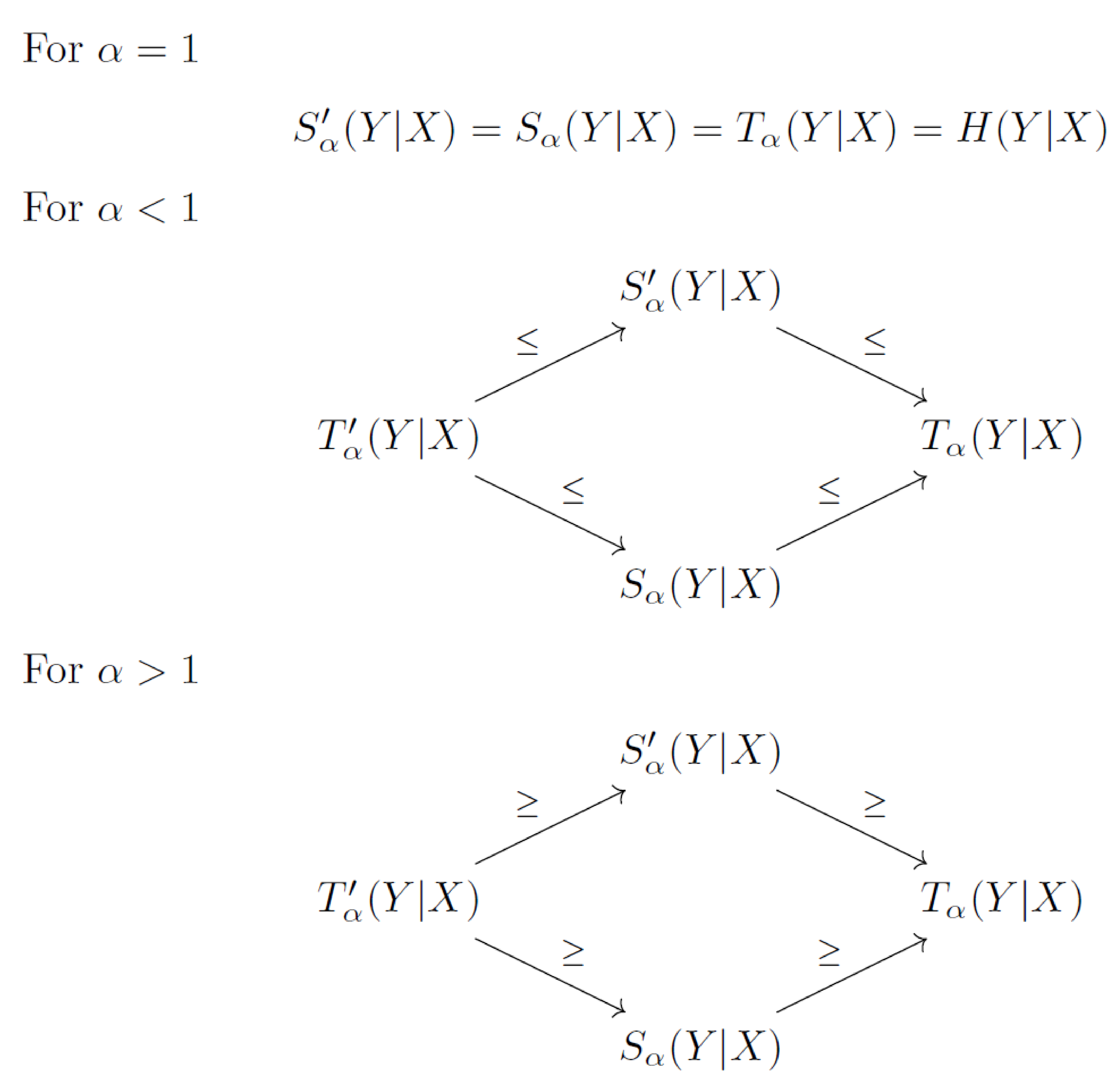

Figure 1: Summary of the relations between the several proposal for the definition of conditional Tsallis entropy.

Conclusions and future work

We can say that none of the proposals fulfil all the properties. The definition $T_\alpha(Y|X)$ is the candidate that fulfils more properties and $S'_\alpha(Y|X)$ is the one that fulfils fewer properties. For future work, since all the definitions focus on possible different aspects of the entropy it would be important to consider a deeper study in this area and its possible applications, aiming to develop theory that would emphasize the best proposal for each area or eventually presenting an ultimate version for the conditional Tsallis entropy that would satisfy all the desirable properties.

References

[1] F. Shigeru. Information theoretical properties of Tsallis entropies. Journal of Mathematical Physics, 47(2), 2006.
[2] S. Manije, G. Borzadaran, and M. Dehak. Conditional Tsallis entropy. Cybernetics and Information Technologies, 13:37-42, 05 2013.

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