Gauge freedom of entropies on q-Gaussian distributions

Hiroshi Matsuzoe

Nagoya Institute of Technology, Nagoya, Japan

Entropy 2021: The Scientific Tool of the 21st Century, from May 5 to 7, 2021.

This is a joint work with Asuka Takatsu at Tokyo Metropolitan University.

A q-Gaussian distribution is a generalization of Gaussian distributions. For a fixed $q \in [1, 3)$, the set of all q-Gaussian distributions admits information geometric structures such as an entropy, a divergence and a Fisher metric via escort expectations. The ordinary expectation of a random variable is the integral of the random variable with respect to its probability distribution. Escort expectations admit us to replace the law to any other distributions. A choice of escort expectations on the set of all q-Gaussian distributions determines an entropy and a divergence. The q-escort expectation is one of most important expectations since this determines the Tsallis entropy and the α-divergence.

The phenomenon gauge freedom of entropies is that different escort expectations determine the same entropy, but different divergences.

In this presentation, we first introduce a refinement of the q-logarithmic function. Then we demonstrate the phenomenon on an open set of all q-Gaussian distributions by using the refined q-logarithmic functions. We write down the corresponding Riemannian metric.

1 q-Logarithmic functions and their refinements

For $q \in \mathbb{R}$, we set $\chi_q(s) := s^q$. The *q-logarithmic function* is defined by

$$\ln_q(t) := \int_1^t \frac{1}{\chi_q(s)} ds.$$

The inverse function of $\ln_q$ is called the q-exponential function.

(1) For $q = 1$, we have that

$$\ln_1(t) = \log(t) \qquad \text{for } t \in (0, \infty),$$
$$\exp_1(\tau) = \exp(\tau). \qquad \text{for } \tau \in \mathbb{R}.$$

(2) For $q \neq 1$, we have that

$$\ln_q(t) = \frac{t^{1-q} - 1}{1 - q} \qquad \text{for } t \in (0, \infty),$$
$$\exp_q(\tau) = \{1 + (1 - q)\tau\}^{\frac{1}{1-q}} \qquad \text{for } \tau \in \ln_q(0, \infty).$$

Refined deformation function

For $q \in \mathbb{R}$ and $a \in \mathbb{R} \setminus \{0\}$, define a function $\chi_{q,a} : (0, 1) \to (0, \infty)$ by

$$\chi_{q,a}(s) := \chi_q(s) \cdot (-\ln_q(s))^{1-a}.$$

Refined logarithm, Refined exponential

We can then define the a-refined q-logarithmic function $\ln_{q,a}$ and the a-refined q-exponential function $\exp_{q,a}$ by

$$\ln_{q,a}(t) := -\frac{1}{a}\left(-\ln_q(t)\right)^a, \qquad \text{for } t \in (0, 1),$$
$$\exp_{q,a}(\tau) = \exp_q\left(-(-a\tau)^{\frac{1}{a}}\right), \qquad \text{for } \tau \in \ln_{q,a}(0, 1). \tag{1}$$

Lemma 1.1 *Let $q \in \mathbb{R}$ and $a \in \mathbb{R} \setminus \{0\}$. For $\tau \in \ln_{q,a}(0, 1)$, we have that*

$$\frac{d^n}{d\tau^n} \exp_{q,a}(\tau) = \exp_{q,a}(\tau)^{(n-1)(q-1)+q}(-a\tau)^{\frac{n(1-a)}{a}} \sum_{j=0}^{n-1} b_j^n(q, a) \cdot (-a\tau)^{-\frac{j}{a}},$$

where $\{b_j^n = b_j^n(q, a)\}_{n\in\mathbb{N}, 0\le j\le n-1}$ satisfies

$$b_0^1 = 1,$$
$$b_j^{n+1} = \begin{cases} \{na(q-1) + 1\}b_0^n & \text{if } j = 0, \\ \{(na + j)(q - 1) + 1\}b_j^n - \{n(1 - a) - (j - 1)\}b_{j-1}^n & \text{if } j \neq 0, n, \\ (na - 1)b_{n-1}^n & \text{if } j = n. \end{cases}$$

Remark 1.2 *For $q \in \mathbb{R}$ and $a \in \mathbb{R} \setminus \{0\}$, we have that*

$$b_0^1 = 1, \quad b_0^2 = a(q - 1) + 1, \quad b_0^3 = \{2a(q - 1) + 1\}\{a(q - 1) + 1\},$$
$$b_1^2 = a - 1, \quad b_1^3 = (a - 1)\{(4a + 1)(q - 1) + 3\},$$
$$b_2^3 = (a - 1)(2a - 1).$$

Corollary 1.3 *For $a \in \mathbb{R} \setminus \{0\}$ and $n \in \mathbb{N}$, then $b_0^n(1, a) = 1$.*

Corollary 1.4 *Let $q \in \mathbb{R}$ and $n \in \mathbb{N}$. For $1 \le j < n$, then $b_j^n(q, 1) = 0$.*

2 Escort expectations

The *escort expectation* of $f \in L^1(\nu)$ with respect to ν on Ω is defined by

$$\mathbb{E}_\nu[f] := \int_\Omega f(\omega) d\nu(\omega). \tag{2}$$

Let $\mathcal{S}$ be a manifold of positive probability densities on a measure space $(\Omega, \mathfrak{m})$. Take $T \in (0, \infty]$ such that $T > \sup\{p(\omega) \mid p \in \mathcal{S},\ \omega \in \Omega\}$.

Let $\ell : (0, T) \to \mathbb{R}$ be a differentiable function such that $\ell' > 0$ in $(0, T)$. For $p \in \mathcal{S}$, we define a measure $\nu_{\ell;p}$ on Ω as the absolutely continuous measure with respect to $\mathfrak{m}$ with Radon–Nikodym derivative

$$\frac{d\nu_{\ell;p}}{d\mathfrak{m}}(\omega) := \frac{1}{\ell'(p(\omega))}.$$

We remark that, in the case $\ell = \log$, the escort expectation (2) is nothing but the ordinary expectation.

3 Gauge freedom of Entropies

A choice of differentiable functions ℓ determines an entropy and a relative entropy on $\mathcal{S}$. The phenomenon gauge freedom of entropies is that different escort expectations determine the same entropy, but different relative entropies. In this section, we demonstrate gauge freedom of entropies on the set of q-Gaussian distributions over $\mathbb{R}$ for $1 \le q < 3$.

q-Gaussian distribution

For $1 \le q < 3$ and $\xi = (\mu, \sigma) \in \mathbb{R} \times (0, \infty)$, the *$q$-Gaussian measure* with location parameter μ and scale parameter σ on $\mathbb{R}$ is

$$p_q(x; \xi) = p_q(x; \mu, \sigma) := \frac{1}{Z_q\sigma} \exp_q\left(-\frac{1}{3 - q}\left(\frac{x - \mu}{\sigma}\right)^2\right).$$

We call $p_q(x; \xi) = p_q(x; \mu, \sigma)$ the q-Gaussian distribution with location parameter μ and scale parameter σ.

Set $\Sigma_q = \{\sigma > 0 \mid 1/(Z_q\sigma) < 1\}$. For $1 \le q < 3$, $a \in \mathbb{R} \setminus \{0\}$ and $\xi \in \mathbb{R} \times \Sigma_q$, we define a measure $\nu_{q,a;\xi}$ on $\mathbb{R}$ by

$$\frac{d\nu_{q,a;\xi}}{dx}(x) := \frac{1}{\ln'_{q,a}(p_q(x; \xi))}.$$

Take $\xi \in \mathbb{R} \times \Sigma_q$ and set $p = p_q(\cdot; \xi) \in \mathcal{S}_q$.

(1) The *(q, a)-cross entropy* of p with respect to $r \in \mathcal{S}_q$ is defined by

$$d_{q,a}(p, r) := -\mathbb{E}_{\nu_{q,a;\xi}}[\ln_{q,a}(r)].$$

(2) The *(q, a)-entropy* of p is defined by

$$\mathrm{Ent}_{q,a}(p) := d_{q,a}(p, p).$$

(3) The *(q, a)-relative entropy* of p with respect to $r \in \mathcal{S}_q$ is defined by

$$D^{(q,a)}(p, r) := -d_{q,a}(p, p) + d_{q,a}(p, r).$$

Remark 3.1 *The $(q, 1)$-entropy coincides with the* Boltzmann–Shannon entropy *if $q = 1$, and the* Tsallis entropy *otherwise.*

Gauge freedom of entropies

Theorem 1 *Let $1 \le q < 3$ and $a \in \mathbb{R} \setminus \{0\}$. Then*

$$\mathrm{Ent}_{q,1} = a\mathrm{Ent}_{q,a},$$
$$D^{(q,1)} \neq \lambda D^{(q,a)}, \quad d_{q,1} \neq \lambda d_{q,a} \qquad \textit{for } a \neq 1 \textit{ and } \lambda \in \mathbb{R}.$$

4 Refined Riemannian metrics

For $1 \le q < 3$, $a \in \mathbb{R} \setminus \{0\}$, set

$$\Sigma_{q,a} := \left\{\sigma \in \Sigma_q \;\middle|\; Z_q\sigma > \exp_q\left(\max\left\{0, \frac{1 - a}{q}\right\}\right)\right\},$$
$$\mathcal{S}_{q,a} := \{p_q(\cdot; \xi) \in \mathcal{S}_q \mid \xi \in \mathbb{R} \times \Sigma_{q,a}\}.$$

Theorem 2 *For $\xi \in \mathbb{R} \times \Sigma_{q,a}$ and $s, t \in \{\mu, \sigma\}$,*

$$g^{(q,a)}\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right)(p_q(\cdot; \xi)) := g_{st}^{(q,a)}(\xi)$$
$$g_{st}^{(q,a)}(\xi) := \int_{\mathbb{R}} \frac{\partial}{\partial s} \ln_{q,a}(p_q(x; \xi)) \cdot \frac{\partial}{\partial t} \ln_{q,a}(p_q(x; \xi)) \times \exp''_{q,a}(\ln_{q,a}(p_q(x; \xi)))\, dx.$$

determines a Riemannian metric on $\mathcal{S}_{q,a}$.

We compute the exact values:

$$g_{\mu\mu}^{(q,a)}(\xi) = \frac{4}{(3 - q)^2} \sum_{j=0}^{1} \frac{b_j^2}{(Z_q\sigma)^{2(1-q)}\sigma^2} \int_{\mathbb{R}} \left(\frac{x - \mu}{\sigma}\right)^2 \frac{p_q(x; \xi)^{2q-1}}{(-\ell_q(x, \xi))^j} dx$$
$$= \frac{4}{(3 - q)^2} \sum_{j=0}^{1} \frac{b_j^2}{(Z_q\sigma)^{2(1-q)}\sigma^2} \Phi(q, 2, 1, j; \xi),$$
$$g_{\sigma\sigma}^{(q,a)}(\xi) = \sum_{j=0}^{1} \frac{b_j^2}{(Z_q\sigma)^{2(1-q)}\sigma^2} \int_{\mathbb{R}} \left\{1 - \left(\frac{x - \mu}{\sigma}\right)^2\right\}^2 \frac{p_q(x; \xi)^{2q-1}}{(-\ell_q(x, \xi))^j} dx$$
$$= \sum_{j=0}^{1} \frac{b_j^2}{(Z_q\sigma)^{2(1-q)}\sigma^2} \sum_{k=0}^{2} \binom{2}{k} (-1)^k \Phi(q, 2, k, j; \xi),$$

for $\xi \in \mathbb{R} \times \Sigma_{q,a}$, where we set

$$\Phi(q, n, k, j; \xi) := \int_{\mathbb{R}} \left(\frac{x - \mu}{\sigma}\right)^{2k} \frac{p_q(x; \xi)^{(n-1)(q-1)+q}}{(-\ell_q(x, \xi))^j} dx.$$

Proposition 4.1 *For $a = 1$ and $\xi = (\mu, \sigma) \in \mathbb{R} \times \Sigma_{q,a}$, we have that*

$$g_{\mu\mu}^{(q,1)}(\xi) = \frac{1}{\sigma^2}, \qquad g_{\sigma\sigma}^{(q,1)}(\xi) = \frac{3 - q}{\sigma^2}.$$

This implies that $(\mathcal{S}_{q,a}, g^{(q,1)})$ is a space of constant curvature.

References

[1] Hiroshi Matsuzoe and Asuka Takatsu, *Gauge Freedom of Entropies on q-Gaussian Measures*, In: Progress in Information Geometry, Signals and Communication Technology. Springer, Cham, (2021), 127–152.