A Dual Measure of Uncertainty: The Deng Extropy

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Entropy and Extropy

Let X be a discrete random variable with support $\{x_1, \dots, x_n\}$ and with probability mass function vector $\underline{p} = (p_1, \dots, p_n)$. The Shannon entropy and the extropy of X are defined as

$$H(X) = H(\underline{p}) = -\sum_{i=1}^{n} p_i \log p_i,$$

$$J(X) = J(\underline{p}) = -\sum_{i=1}^{n} (1-p_i)\log(1-p_i).$$

Lad, Sanfilippo and Agrò [4] proved the following property related to the sum of the entropy and the extropy:

$$H(\underline{p}) + J(\underline{p}) = \sum_{i=1}^{n} H(p_i, 1-p_i) = \sum_{i=1}^{n} J(p_i, 1-p_i), \tag{1}$$

where $H(p_i, 1-p_i) = J(p_i, 1-p_i) = -p_i \log p_i - (1-p_i)\log(1-p_i)$ are the entropy and the extropy of a discrete random variable which support has cardinality two and probability mass function vector is $(p_i, 1-p_i)$.

Dempster–Shafer theory of evidence

Let X be a frame of discernment, i.e., a set of mutually exclusive and collectively exhaustive events indicated by $X = \{\theta_1, \theta_2, \dots, \theta_{|X|}\}$. The power set of X is indicated by 2^X. A function $m : 2^X \to [0,1]$ is called a mass function or a basic probability assignment (BPA) if

$$m(\emptyset) = 0 \text{ and } \sum_{A \in 2^X} m(A) = 1.$$

Deng Entropy and Deng Extropy

Deng entropy [2] is defined as

$$E_d(m) = -\sum_{A \subseteq X : m(A) > 0} m(A) \log_2 \left(\frac{m(A)}{2^{|A|} - 1} \right).$$

In order to preserve an analogous of (1) we define (Buono and Longobardi [1]) the Deng extropy as

$$EX_d(m) = -\sum_{A \subset X : m(A) > 0} (1 - m(A)) \log_2 \left(\frac{1 - m(A)}{2^{|A^c|} - 1} \right).$$

Proposition 1. *Let m be a BPA for a frame of discernment X. Then*

$$\begin{aligned} E_d(m) + EX_d(m) &= \sum_{A \subset X : m(A) > 0} E_d(m_A^*) - m(X) \log_2 \left(\frac{m(X)}{2^{|X|} - 1} \right) \\ &= \sum_{A \subset X : m(A) > 0} EX_d(m_A^*) - m(X) \log_2 \left(\frac{m(X)}{2^{|X|} - 1} \right), \end{aligned}$$

where m_A^* is a BPA on X defined as

$$m_A^*(B) = \begin{cases} m(A), & \text{if } B = A \\ 1 - m(A), & \text{if } B = A^c \\ 0, & \text{otherwise.} \end{cases}$$

Example 1. *Given a frame of discernment X with cardinality n and a BPA m such that $m(i) = \frac{1}{n}$, for $i = 1, \dots, n$, we have*

$$\begin{aligned} E_d(m) &= \log_2(n), \\ EX_d(m) &= (n-1) \left[\log_2 \left(\frac{n}{n-1} \right) + \log_2(2^{n-1} - 1) \right]. \end{aligned}$$

Example 2. *Given a frame of discernment $X = \{1, 2, \dots, 15\}$ and a BPA m such that $m(3,4,5) = 0.05$, $m(6) = 0.05$, m(A)=0.8, $m(X) = 0.1$. When A changes, the values for the Deng extropy and entropy are obtained in Table 1.*

A	Deng Extropy	Deng Entropy
$\{1\}$	28.104	2.6623
$\{1,2\}$	27.904	3.9303
$\{1,2,3\}$	27.704	4.9082
$\{1,\dots,4\}$	27.504	5.7878
$\{1,\dots,5\}$	27.304	6.6256
$\{1,\dots,6\}$	27.104	7.4441
$\{1,\dots,7\}$	26.903	8.2532
$\{1,\dots,8\}$	26.702	9.0578
$\{1,\dots,9\}$	26.500	9.8600
$\{1,\dots,10\}$	26.295	10.661
$\{1,\dots,11\}$	26.086	11.462
$\{1,\dots,12\}$	25.866	12.262
$\{1,\dots,13\}$	25.621	13.062
$\{1,\dots,14\}$	25.304	13.862

Table 1: *The values of the Deng extropy and the Deng entropy as A changes.*

The Maximum Deng Extropy

Theorem 1. *Let m be a BPA for a frame of discernment X. The maximum Deng extropy for fixed values of $m(X)$ and N number of focal elements different from X, $N = |\mathcal{N}| = |\{A \subset X : m(A) > 0\}|$, is attained if and only if*

$$m(A) = 1 - \frac{N - (1 - m(X))}{\sum_{B \in \mathcal{N}} (2^{|B^c|} - 1)} (2^{|A^c|} - 1), \quad A \in \mathcal{N}.$$

In this case, the value of the maximum Deng extropy is

$$EX_d^* = -[N - (1 - m(X))] \log_2 \frac{N - (1 - m(X))}{\sum_{A \in \mathcal{N}} (2^{|A^c|} - 1)}.$$

Application to Pattern Recognition

The dataset Iris is composed of 150 samples. For each one, we have the sepal length (SL), the sepal width (SW), the petal length (PL), the petal width (PW) and we want to investigate about the class that is only one between Iris Setosa (Se), Iris Versicolour (Ve) and Iris Virginica (Vi). We use sample of max–min value to generate a model of interval numbers. Suppose the selected sample data are (6.1, 3.0, 4.9, 1.8, Iris Virginica).

Item	SL	SW	PL	PW
Se	[4.4,5.8]	[2.3,4.4]	[1.0,1.9]	[0.1,0.6]
Ve	[4.9,7.0]	[2.0,3.4]	[3.0,5.1]	[1.0,1.7]
Vi	[4.9,7.9]	[2.2,3.8]	[4.5,6.9]	[1.4,2.5]
Se, Ve	[4.9,5.8]	[2.3,3.4]	–	–
Se, Vi	[4.9,5.8]	[2.3,3.8]	–	–
Ve, Vi	[4.9,7.0]	[2.2,3.4]	[4.5,5.1]	[1.4,1.7]
Se, Ve, Vi	[4.9,5.8]	[2.3,3.4]	–	–

Table 2: The interval numbers of the statistical model.

Four BPAs are generated with a method proposed by Kang et al. [3] based on the similarity of interval numbers. In particular, we evaluate the similarities between the intervals given in Table 2 and singletons of the items of the selected sample. For each one of the four properties, we get seven values of similarity and then we get a BPA by normalizing them. We obtain a combined BPA by using the Dempster rule of Combination.

Item	SL	SW	PL	PW	Combined BPA
$m(Se)$	0.1098	0.1018	0.0625	0.1004	0.0059
$m(Ve)$	0.1703	0.1303	0.1839	0.2399	0.4664
$m(Vi)$	0.1257	0.1385	0.1819	0.3017	0.4656
$m(Se, Ve)$	0.1413	0.1663	0.0000	0.0000	0.0000
$m(Se, Vi)$	0.1413	0.1441	0.0000	0.0000	0.0000
$m(Ve, Vi)$	0.1703	0.1527	0.5719	0.3580	0.0620
$m(Se, Ve, Vi)$	0.1413	0.1663	0.0000	0.0000	0.0000
Deng extropy	5.2548	5.2806	5.1636	4.9477	

Table 3: BPAs based on Kang's method, Deng extropy and final fusion result.

The type of unknown sample is determined by combined BPA. We evaluate the Pignistic Probability Transformation (PPT) of focal elements, which represents a point estimate of belief and can be determined as

$$PPT(A) = \sum_{B : A \subseteq B} \frac{m(B)}{|B|}.$$

We get the maximum value of PPT and Kang's method assigns to the sample the type Iris Versicolour without making the right decision.

We use the Deng extropies to evaluate the weight of each property. For the sample property sepal length we have

$$\omega(SL) = \frac{e^{-EX_d(SL)}}{e^{-EX_d(SL)} + e^{-EX_d(SW)} + e^{-EX_d(PL)} + e^{-EX_d(PW)}}.$$

We use the weights as discounting coefficients to generate new BPAs.

Item	SL	SW	PL	PW	Combined BPA
$m(Se)$	0.0808	0.0730	0.0504	0.1004	0.0224
$m(Ve)$	0.1252	0.0934	0.1482	0.2399	0.4406
$m(Vi)$	0.0925	0.0993	0.1465	0.3017	0.4451
$m(Se, Ve)$	0.1039	0.1192	0.0000	0.0000	0.0000
$m(Se, Vi)$	0.1039	0.1033	0.0000	0.0000	0.0000
$m(Ve, Vi)$	0.1252	0.1095	0.4608	0.3580	0.0919
$m(Se, Ve, Vi)$	0.3684	0.4023	0.1942	0.0000	0.0000

Table 4: The modified BPAs based on the Deng extropy and final fusion result.

Finally, we tested all 150 samples and we get that the global recognition rate of Kang's method is 93.33% whereas the global recognition of the method based on the Deng extropy is 94%.

Item	Setosa	Versicolor	Virginica	Global
Kang's method	100%	96%	84%	93.33%
Method based on EX_d	100%	96%	86%	94%

Table 5: The recognition rate.

References

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