Fisher information of Landau states and relative information against the lowest level

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Abstract

Fisher information of the radial densities for the Landau states specified by two quantum numbers are studied. It increases linearly with the principal quantum number, whereas it decreases monotonically with the azimuthal quantum number m. The relative Fisher information of these states against the lowest Landau level state is calculated. When the reference density with m = 0 is set, it is proportional to the principal quantum number.

Introduction

The wavefunction of electrons in a uniform magnetic field is described by the Landau states and the eigen-energy is quantized. Under the Landau gauge for the vector potential and using the cylindrical polar coordinates (ρ, ϕ, z), The wavefunction is given as [1]

$$\psi_{n_\rho,m}(\rho,\phi,z) = \frac{1}{\sqrt{2\pi}} R_{n_\rho,m}(\rho) e^{im\phi} e^{i\frac{p_z z}{\hbar}}$$

where the radial wavefunction

$$R_{n_\rho,m}(\rho) = N_{n_\rho,m} \exp(-\frac{\rho^2}{4a_H^2}) \rho^{|m|} F(-n_\rho, |m|+1, \frac{\rho^2}{2a_H^2})$$

$N_{n_\rho,m} = \frac{1}{a_H^{|m|+1}|m|!}\left(\frac{(|m|+n_\rho)!}{2^{|m|}n_\rho!}\right)^{\frac{1}{2}}$: Normalization factor

$F(-n,\alpha+1,x) = \frac{n!\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)} L_n^{(\alpha)}(x)$: confluent hypergeometric function

$L_n^{(\alpha)}(x)$: generalized Laguerre polynomial

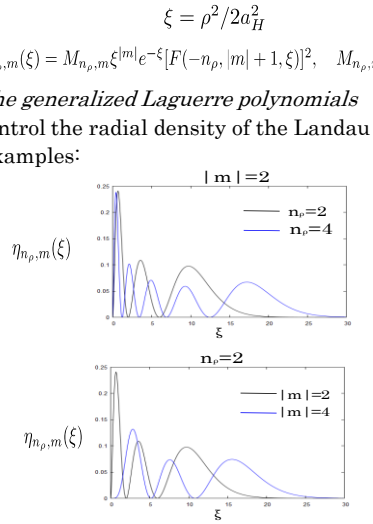
$a_H = \sqrt{\frac{\hbar c}{|e|H}}$: magnetic length

By rescaling, we define the radial probability density from the wavefunction as follows:

$$\xi = \rho^2/2a_H^2$$

$$\eta_{n_\rho,m}(\xi) = M_{n_\rho,m}\xi^{|m|}e^{-\xi}[F(-n_\rho,|m|+1,\xi)]^2, \quad M_{n_\rho,m} = \frac{(|m|+n_\rho)!}{(|m|!)^2 n_\rho!}$$

The generalized Laguerre polynomials $L_{n_\rho}^{(|m|)}\left(\frac{\rho^2}{2a_H^2}\right)$ control the radial density of the Landau states. Examples:



(Upper panel): Broadly spread and nodal densities for larger principal quantum numbers.
(Lower panel): Broadly spread densities for larger azimuthal quantum numbers.

Fisher information

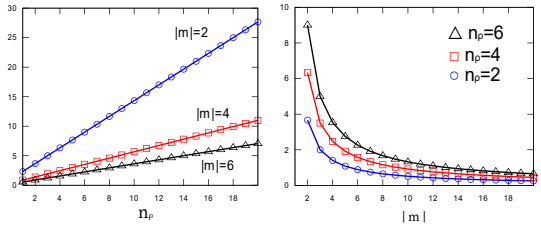
In information theory, Fisher information is defined:

$$I(\rho) := \int \frac{\|\nabla\rho\|^2}{\rho} dx$$

for a probability density function ρ. Note that this is a particular case of the original by Fisher [2] when ρ satisfies the shift invariance property.

Fisher information is more suitable than Shannon entropy to capture and quantify the gradient (nodal) structure. Large values mean a stronger localization. Thus, we evaluate:

$$I(\eta_{n_\rho,m}) = \int_0^\infty \frac{1}{\eta_{n_\rho,m}(\xi)} \left(\frac{d\eta_{n_\rho,m}(\xi)}{d\xi}\right)^2 d\xi$$



Fisher information linearly increases with the principal quantum number n, whereas it decreases with | m |. This behavior contrasts to the case of the hydrogen-like atoms, in which $I(\rho)$ decreases with n [2].

Relative Fisher information (RFI)

The RFI of the probability density function f(x) with respect to g(x) is defined:

$$I(f|g) := \int \|\nabla \log \frac{f(x)}{g(x)}\|^2 f(x) dx$$

The lowest Landau state with $n_\rho = 0$ can be set as the reference probability density function g(x).

$$I(\eta_{n_\rho,m}(\xi)|\eta_{0,m}(\xi)) = \int_0^\infty \eta_{n_\rho,m}(\xi)\left|\frac{d}{d\xi}\log\left(\frac{\eta_{n_\rho,m}(\xi)}{\eta_{0,m}(\xi)}\right)\right|^2 d\xi$$

where $\eta_{0,m}(\xi) = \frac{\xi^{|m|}e^{-\xi}}{|m|!}$.

Using the following properties,

$$L_0^{(\alpha)}(\xi) = 1 \qquad F(0,\alpha+1,\xi) = 1$$

$[L_n^{(\alpha)}(\xi)]' = -L_{n-1}^{(\alpha+1)}(\xi)$: (Differential relation)

$$\int_0^\infty \xi^\alpha e^{-\xi} L_n^{(\alpha)}(\xi) L_m^{(\alpha)}(\xi) d\xi = \frac{\Gamma(\alpha+n+1)}{n!}\delta_{m,n}$$:(Orthogonality relation)

$$F'(-n_\rho,|m|+1,\xi) = -\frac{\Gamma(|m|+1)n_\rho!}{\Gamma(n_\rho+|m|+1)} L_{n_\rho-1}^{(|m|+1)}(\xi)$$

We find that RFI is proportional to the principal quantum number:

$$I(\eta_{n_\rho,m}(\xi)|\eta_{0,m}(\xi)) = 4n_\rho$$

It is interesting that | m | is irrelevant when the radial densities with the same azimuthal quantum number as the reference are compared.

Alternative reference state

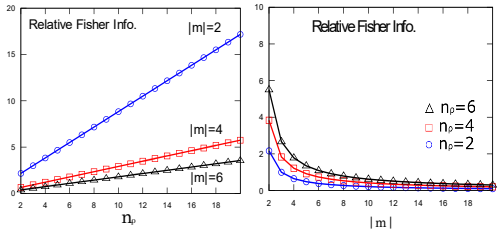
When we choose the reference state as n=0, m=0, i.e., $\eta_{0,0}(\xi) = e^{-\xi}$, the calculation is tractable:

$$I(\eta_{n_\rho,m}(\xi)|\eta_{0,0}(\xi)) = \frac{n_\rho!}{(n_\rho+|m|)!}(I_1+I_2+I_3)$$

where the integrals are defined as

$$I_1 = |m|^2 \int_0^\infty \xi^{|m|-2} e^{-\xi} [L_{n_\rho}^{(|m|)}(\xi)]^2 d\xi$$
$$I_2 = -4|m| \int_0^\infty \xi^{|m|-1} e^{-\xi} L_{n_\rho}^{(|m|)}(\xi) L_{n_\rho-1}^{(|m|+1)}(\xi) d\xi$$
$$I_3 = 4\int_0^\infty \xi^{|m|} e^{-\xi} [L_{n_\rho-1}^{(|m|+1)}(\xi)]^2 d\xi.$$

For the details of the evaluation of these integrals, see [4], and [5] for the integral formula involving the product of two Laguerre polynomials with different integer degrees and orders. The behaviour of RFI as a function of n and | m | are similar to the Fisher info. (middle column).



Contrary to the use of the reference state (n=0, m), RFI depends on both quantum numbers. The high | m | states are closer to the lowest Landau state radial density (i.e., exponential decay).

This dependency is remarkably analogous to the case of the D-dimensional isotropic quantum oscillator, in which RFI of the radial wavefunction depends on both n and l (orbital quantum #) [6].

Summary

As a descriptive indicator of the radial spreading of the wavefunction of *free-electron* Landau states, we computed Fisher information associated with the radial density. It shows linearity in the principal quantum number. RFI is more involved in computation and depends on the reference states. In this study, we compared two choices of the lowest Landau states and showed these behaviors.

References

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