On the relation of Eckart and Landau-Lifshitz reference frames for higher orders in the dissipative fluxes couplings

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Abstract

In the kinetic description of a relativistic gas, macroscopic quantities are usually studied either in the reference frame that moves with the fluid (the so-called Eckart frame) or in the frame with no energy dissipation flux (Landau-Lifshitz frame). The Landau-Lifshitz frame requires several approximations to relate it with the Eckart's or particle frame in a closer and detailed view. For the energy-momentum tensor not to contain energy fluxes is necessary to neglect couplings among dissipative flows. It is well known that at first order, the particle 4-flux contains dissipative term related to the heat flux and also defines a Lorentz reference frame through a timelike vector. In this work, we relate these reference frames to higher orders in dissipative fluxes couplings and compare the properties of the corresponding systems of transport equations in both frames.

1. Motivation and introduction

- Distribution function: $f\left(x^{\mu}, v^{\mu}, t\right) \leftarrow$ number of molecules at time t in volume $dv^{\mu}dr_{\mu}$
- Average $u^{\mu} = \frac{1}{n}\int v^{\mu} f d^{*}v$ (hydrodynamic velocity) define a preferential direction.
- In the particles frame, we used 3+1 descomposition

$$u^{a} \qquad \text{temporal direction}$$

$$h^{ab} = \eta^{\mu\nu} - \frac{u^{\mu}u^{\nu}}{c^{2}} \qquad \text{space hypersurface}$$

- In this represetation we obtained

$$N^{\mu} = \int v^{\mu} f d^{*}v = n u^{\mu},$$

$$T^{\mu\nu} = \int v^{\mu}v^{\nu} f d^{*}v = \frac{n\varepsilon}{c^{2}}u^{\mu}u^{\nu} + p h^{\mu\nu} + \frac{q^{\mu}}{c^{2}}u^{\nu} + \frac{q^{\nu}}{c^{2}}u^{\mu} + \pi^{\mu\nu}$$

$$\varepsilon = mc^{2}\int \gamma_{k}^{2} f d^{*}v, \; q^{\mu} = m h^{\mu\nu}\int \gamma_{k} v_{\nu} f d^{*}v,$$

$$\pi^{\mu\nu} - p h^{\mu\nu} = h^{\mu\eta}h^{\nu\beta}\int v_{\eta}v_{\beta} f d^{*}v.$$

- The evolution equation for$f\left(x^{\mu}, v^{\mu}, t\right)$ is given by any average using the Boltzmann equation:

$$v^{\mu} f_{,\mu} = \mathcal{C}\left(f\right) \leftarrow \text{for a molecule with speed}$$

$$\int \psi\left(v^{\nu}\right) v^{\mu} f_{,\mu} d^{*}v = \int \psi\left(v^{\nu}\right)\mathcal{C}\left(f\right) d^{*}v \leftarrow \text{for a gas}$$

- If $\psi\left(v^{\nu}\right)$ is the colisional invariant,

$$\int \psi\left(v^{\nu}\right) v^{\mu} f_{,\mu} d^{*}v = 0 \begin{cases} \nearrow N^{\mu}_{,\mu} = 0 \Rightarrow \text{continuity eq.} \\ \searrow T^{\mu\nu}_{,\nu} = 0 \Rightarrow \text{energy-momentum eq.} \end{cases}$$

- The last equations could present the generic instability if we used q^{μ} with $\dot{u}^{\mu}$. We could use another frame called the energy frame. In this frame, the generic instability does not occur. The phenomenology of this framework is

$$N^{\mu} = n_{L}w^{\mu} + J^{\mu}$$

$$T^{\mu\nu} = \frac{n_{L}\epsilon}{c^{2}}w^{\mu}w^{\nu} + p_{L}H^{\mu\nu} + \Pi^{\mu\nu}$$

What is the physical interpretation for the energy framework, and which are the differences of the balance equations between the particle and energy frameworks?
Are there any equation equivalent in the no relativistic case?

2. No Relativistic Case

2.1 Particle Frame

- From Boltzmann equation we get

$$\frac{dn}{dt} + n\frac{\partial u^{a}}{\partial x^{a}} = 0, \; mn\frac{du_{b}}{dt} + \frac{\partial \pi^{a}_{b}}{\partial x^{a}} = 0,$$

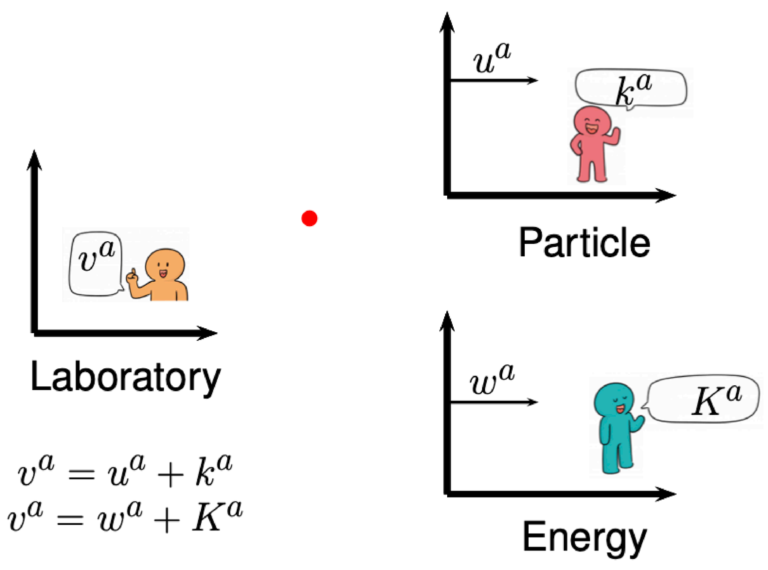
$$n\frac{d\varepsilon}{dt} + \pi^{ab}\frac{\partial u_{b}}{\partial x^{a}} + \frac{\partial q^{a}}{\partial x^{a}} = 0$$

$$\varepsilon = \int \frac{mk^{2}}{2} f d^{*}v,$$

$$\pi^{ab} = \int \left(mk^{a}\right) k^{b} f d^{*}v,$$

$$q^{a} = \int \frac{mk^{2}}{2}k^{a} f d^{*}v, \; \boxed{v^{a} = u^{a} + k^{a}}.$$

- We are looking at a frame where the heat flux vanishes or a relationship where there is no thermal dissipation.



$$v^{a} = u^{a} + k^{a}$$
$$v^{a} = w^{a} + K^{a}$$

2.2 Energy Frame

- In the energy frame the particle flux is given by

$$nJ_{b} = \int K_{b} f d^{3}v \rightarrow J_{b} = k_{b} - K_{b}$$

$$nu_{b} = \int v_{b} f d^{3}v \rightarrow u_{b} = w_{b} + J_{b}$$

2.3 Heat flux equal to zero

- Transport equations in the new frame:

$$n\epsilon \equiv \int \left(\frac{m}{2}K_{b}K^{b}\right) f d^{3}v \rightarrow \boxed{n\epsilon = \frac{m}{2}nJ^{2} + n\varepsilon}$$

$$nQ_{a} \equiv \int \left(\frac{m}{2}K_{b}K^{b}\right) K_{a} f d^{3}v$$

$$\rightarrow \boxed{Q_{a} = J_{a}\left(\frac{mn}{2}J^{2} + n\varepsilon\right) + J_{b}\pi^{b}_{a} + q_{a}}$$

$$\Pi^{ab} = \int \left(mK^{a}\right) K^{b} f d^{3}v \rightarrow \boxed{\Pi^{ab} = \pi^{ab} + mnJ^{a}J^{b}}$$

- The heat flux vanishes in non-viscous fluid if:

$$\boxed{J_{a} = -\frac{q_{a}}{n\epsilon}}$$

2.4 Equation for a heat flux

- In this frame the transport equations are:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + w^{a}\frac{\partial}{\partial x^{a}}$$

$$\frac{dn}{dt} + n\frac{\partial w^{a}}{\partial x^{a}} = -n\frac{\partial J^{a}}{\partial x^{a}} - J^{a}\frac{\partial n}{\partial x^{a}}$$

$$mn\frac{dw_{b}}{dt} + \frac{\partial \Pi^{a}_{b}}{\partial x^{a}} = -mn\frac{dJ_{b}}{dt} + mJ_{b}\frac{\partial}{\partial x^{a}}\left(nJ^{a}\right) - mnJ^{a}\frac{\partial w_{b}}{\partial x^{a}}$$

$$n\frac{d\varepsilon}{dt} + \Pi^{ba}\frac{\partial w^{b}}{\partial x^{a}} = mnJ_{b}J^{a}\frac{\partial w^{b}}{\partial x^{a}} - nJ^{a}\frac{\partial \varepsilon}{\partial x^{a}} - \left(\Pi^{ab} - mnJ^{a}J^{b}\right)\frac{\partial J_{b}}{\partial x^{a}} - \frac{\partial q^{a}}{\partial x^{a}}$$

- Thus there is no thermal dissipation, the new framework must satisfy:

$$mnJ_{b}J^{a}\frac{\partial w^{b}}{\partial x^{a}} - nJ^{a}\frac{\partial \varepsilon}{\partial x^{a}} - \left(\Pi^{ab} - mnJ^{a}J^{b}\right)\frac{\partial J_{b}}{\partial x^{a}} = \frac{\partial q^{a}}{\partial x^{a}}$$

or, neglecting viscosity

$$nJ^{a}\left(J_{b}\frac{\partial u^{b}}{\partial x^{a}} - mn\frac{\partial \varepsilon}{\partial x^{a}}\right) = \frac{\partial q^{a}}{\partial x^{a}}$$

3. Relativistic Case

3.1 Particle frame

$$\text{Eckart} \begin{cases} \nearrow u^{\mu} \; \text{temporal} \\ \longrightarrow h^{\mu\nu} = \eta^{\mu\nu} + \frac{1}{c^{2}}u^{\mu}u^{\nu} \; \text{espatial} \\ \searrow v^{\mu} = h^{\mu\nu}v_{\nu} - \left(\frac{u^{\nu}v_{\nu}}{c^{2}}\right)u^{\mu} = h^{\mu\nu}v_{\nu} + \gamma_{k}u^{\mu} \end{cases}$$

- Particle flow:

$$N^{\mu} = \int \left(h^{\mu\nu}v_{\nu} + \gamma_{k}u^{\mu}\right) f d^{*}v = h^{\mu\nu}u_{\nu} + u^{\mu}\int \gamma_{k} f d^{*}v = nu^{\mu}$$

3.2 Energy Frame

$$\text{Landau} \begin{cases} \nearrow w^{a} \; \text{temporal} \\ \longrightarrow H^{\mu\nu} = \eta^{\mu\nu} + \frac{1}{c^{2}}w^{\mu}w^{\nu} \; \text{espatial} \\ \searrow v^{\mu} = H^{\mu\nu}v_{\nu} - \left(\frac{w^{\nu}v_{\nu}}{c^{2}}\right)w^{\mu} = H^{\mu\nu}v_{\nu} + \gamma_{K}w^{\mu} \end{cases}$$

- Particle flow:

$$N^{\mu} = \int \left(H^{\mu\nu}v_{\nu} + \gamma_{K}w^{\mu}\right) f d^{*}v = H^{\mu\nu}\int v_{\nu} f d^{*}v + w^{\mu}\int \gamma_{K} f d^{*}v = J^{\mu} + n_{L}w^{\mu}$$

- The relation between two frames is

$$J^{\mu} = nH^{\mu\nu}u_{\nu} = nu^{\mu} - n\alpha w^{\mu} \rightarrow \boxed{w^{\mu} = \frac{1}{\alpha}\left(u^{\mu} - \frac{J^{\mu}}{n}\right)},$$

$$\alpha = \frac{n_{L}}{n} = -\frac{u^{\mu}w_{\mu}}{c^{2}}$$

- The energy-momentum tensor is given by

$$T^{\mu\nu} = \frac{n\alpha}{c^{2}}w^{\mu}w^{\nu} + p_{L}h^{\mu\nu} + \frac{1}{c^{2}}w^{\mu}Q^{\nu} + \frac{1}{c^{2}}w^{\nu}Q^{\mu} + \Pi^{\mu\nu}$$

where

$$Q^{\mu} = -\frac{m}{c^{2}}H^{\mu\nu}w^{\lambda}\int v_{\nu}v_{\lambda} f d^{*}v = mH^{\mu\nu}\int \gamma_{K}v_{\nu} f d^{*}v$$

$$\Pi^{\mu\nu} - p_{L}h^{\mu\nu} = H^{\mu\lambda}H^{\nu\eta}\int v_{\lambda}v_{\eta} f d^{*}v$$

- The heat flux is given by

$$Q^{\mu} = H^{\mu\nu}\left(q_{\nu} + n\varepsilon J_{\nu}\right) + \Pi^{\nu\lambda}J_{\lambda}$$

-To obtain the relation between q^{μ} and J^{μ}, it is necessary to neglect the viscous term.
- We obtained the same relation for the non-relativistci case:

$$\boxed{J^{\mu} = \frac{q^{\mu}}{n\varepsilon}}$$

- The balance equation does not contain the heat flux because it is not part of $T^{\mu\nu}$.

4. Conclusions and future work

- It is possible to find a frame where the heat dissipation cancels out both relativistic and non-relativistic by using kinetic theory.
- However, in both scenarios, decoupling the viscous tensor in the equilibrium equations for velocity is a complicated task that requires further analysis or even numerical calculations.
- As future work, it is necessary to complement the formalism by introducing transformations between the three reference systems: laboratory, particle, and energy.
- Obtain transport coefficient for J^{a}

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