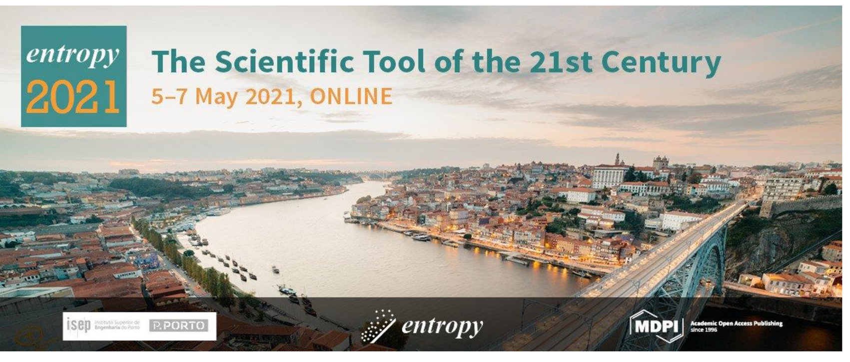


# Comparative performance analysis of a Brownian Carnot cycle from the perspective of a stochastic model against the Linear Irreversible Thermodynamic theory.

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## Abstract

In this work we present a Brownian Carnot cycle, which has already been studied by Schmield et al. (2007) as well as by Izumida and Okuda (2010); but now considering two different working regimes, namely the Maximum Ecological Function (MEC) and the Maximum Efficient Power (MEP). For the MEC and MEP working regimes, the thermodynamic properties of the cycle are obtained, in particular, it is showed that the maximum efficiency now depends on two parameters α and β, instead of only one parameter obtained previously by Schmiled et al. in the maximum power regime. It is worthwhile to notice that for characteristic values of α and β the original results obtained by Schmield are recovered. From the previous observations, the authors consider that the results obtained represent a more general case that includes other working regimes


## Introduction

Diverse criteria of merit have been proposed for the optimization of finite-time engines. For instance, in the Curzon-Ahlborn work [1], maximization of power was proposed to open the perspective of setting up more realistic theoretical bounds on thermodynamics processes that produce entropy. Considering the CA paper as a starting point, many authors began to introduce different objective functions, namely: In the beginning, some authors proposed the entropy production minimization as one of the first criterion [2]. Then in 1991, Angulo introduced the ecological function [3]. Yilmaz et al [4]. studied the efficient power function which considers the effects on the design of heat engines, as the multiplication of power by the cycle efficiency. It is important remark that one of the most astonishing results obtained, is that those thermal engine models show some universality regarding the behavior of the efficiency when it works at the maximum power regime [5], although the analyzed models were different in nature and scale [6-8]. Kedem et al [9], obtained some qualitative predictions confirmed by experimental data. Then, through Linear Irreversible Thermodynamics (LIT). Although traditionally most energy-transfer devices discussed by the Finite Time Thermodynamics (FTT) were macroscopic heat engines, in recent years, isothermal engines with mesoscopic description have been studied, particularly those concerned with thermal fluctuations named Brownian motors. These engines have been studied since the seminal work of Sekimoto et al. [10], where the author studied the thermodynamic properties of Langevin systems that are far out of equilibrium. Of special interest is Schmiedl's work [7] which studies the efficiency of a Brownian motor at maximum power in a similar manner as the macroscopic Carnot cycle, i.e., by two isothermal branches and two adiabatic ones. In their work Schmiedl, et al. found the efficiency at maximum power as; (A). This relation was considered by the authors as a universal relation for this type of engines, which differs form the famous CA efficiency. Although, under proper conditions, when the linear response regime or ''thermodynamic limit'' is considered (ΔT → 0), we could make a relationship between macroscopic and mesoscopic systems. In 2010, Izumida Y. et al. [11] introduced a model for a Brownian Carnot cycle which obtained equivalent results as those determined by Schmiedl et al. [8] but using Linear Irreversible Thermodynamics (LIT) for the maximum power regime, as (B). This paper shows that the Izumida assumption prevails when we consider other working regimes (different from the maximum power regime). Through the stochastic model of Schmiedl, we find the thermodynamic properties in two different regimes: the one with the maximum ecological function and the one with the maximum efficient power. Then, we show how to obtain comparable results using the LIT with the idea postulated by Izumida et al. [11] concerning the cycle period.

$$\eta_{P_{max}} = \frac{\eta_C}{2-\eta_C/2} \quad (A) \qquad P_{max} = \frac{\mu\left(ln\sqrt{w_2/w_1}\right)^2 \Delta T^2}{16\left(\sqrt{w_2}-\sqrt{w_1}\right)^2} \quad (B)$$

where, $w_1$ and $w_2$ are the particle position variances in the finite cycle according with Schmiedl et al[8].

## The Brownian Carnot cycle under the ecological criterion using the stochastic model

The ecological function by Angulo-Brown $E = P - T_C\sigma$ (1)

The power obtained by Schmield $P = \dot{W} = \dfrac{W}{t_1+t_3}$ (2)

where $t_1, t_3$ are the transition times in stochastic model

The entropy production rate during the Brownian Carnot cycle $\dot{\sigma} = \dfrac{\dot{Q}_H}{T_H} - \dfrac{P-\dot{Q}_H}{T_C}$ (3)

sustituting (2) and (3) in (1) yields $E = \dfrac{2W}{t_1+t_3} - \dfrac{\dot{Q}_H \Delta T}{T_H}$ (4)

From the cyclic process the work is given as, $W = -W_1^{irr} - W_3^{irr} - (T_H - T_C)\Delta S$ (5)

where $W_1^{irr}$ is the mean irreversible work defined by Schmield and S the entropy of the Brownian particle. Likewise, the heat uptake during the cycle from heat reservoir at temperature $T_H$ in step 1 can be expressed as;

$$-Q_H = W_1 + \Delta E_1 = T_H\Delta S - W_1^{irr} \quad (6)$$

Sustituting (5) and (6) in (4), yields

$$\mathrm{E} = \frac{2}{t_1+t_3}\left(-W_1^{irr} - -W_3^{irr} + \Delta T\Delta S\right) - \frac{\Delta T}{T_H}\frac{\left(T_H\Delta S - W_1^{irr}\right)}{t_1+t_3} \quad (7)$$

where $W_1^{irr} = A_{irr}/t_i$ is the mean irreversible work, $\Delta T = T_H - T_C$, $\Delta S$ is the entropy during the during adiabatic steps. After optimization (6), we find, that

$$t_1 = \gamma t_3 = \sqrt{\frac{A_{irr}}{C_{irr}}\left(1-\frac{\Delta T}{2T}\right)}\, t_3 \quad (8)$$

sustituting (8) in (7), yields

If we take $\gamma$=1, $\eta = \frac{\dot{W}}{\dot{Q}_H}$ $\quad \eta_C = \frac{\Delta T}{T}$

$$E_{max} = \frac{A_{irr}\left(-2+\frac{\Delta T}{T}\right) - 2\gamma C_{irr} + \Delta T\Delta S\gamma t_3}{\left(1+\frac{1}{\gamma}\right)(\gamma t_3)} \quad (9)$$

if $A_{irr} = C_{irr}$ and we consider $t_1 = t_3$ from (9), then it is possible re-write (9) as

$$E_{max} = \frac{\Delta T^2\Delta S^2}{A_{irr}}\frac{-2+\eta_C}{(8-3\eta_C)^2} = \frac{\Delta T^2\Delta S^2}{32A_{irr}}\left(1+\frac{O(\eta_C)}{4}\right) \quad (10)$$

Using the efficiency definition and sustituting (8), is posible obtain the efficiency at maximum ecological function *E*

$$\eta_{E_{max}} = \frac{\beta\eta_C}{2-\alpha\eta_C} \quad (11)$$

Eq. (11) is remarkably that obtained by Schmield [8], in our case α, β are different $\alpha = \dfrac{-2A_{irr}}{\gamma^\delta}$

$$\beta = \frac{-2A_{irr}}{\gamma^\delta}\frac{\left(1+\frac{\gamma C_{irr}}{A_{irr}}+\frac{\gamma\delta}{A_{irr}}\right)}{(2-\alpha\eta_C)}$$

Finally, if we consider the special case $t_1 = t_3$ from equation (8), is possible re − write (11) as

$$\eta_{E_{max}} = \frac{\frac{6}{4}\eta_C}{2+\frac{1}{4}\eta_C} = \frac{3}{4}\eta_C - \frac{3}{32}\eta_C^2 + O(\eta_C^3) \quad (12)$$

## The Brownian Carnot cycle under the ecological criterion using Linear Irreversible Thermodynamics

As well is known the Power, the Entropy production, and Ecological function, written in terms of Fluxes and generalized forces are given as;

$$P = -J_1X_1T\ ,\ \sigma = J_1X_1 - T\,(J_1X_1 - J_2X_2)\ \&\ E = -2TL_{11}X_1^2 - 3TL_{12}X_1X_2 - TL_{22}X_2^2$$

If we do $\frac{\partial E}{\partial X_1} = 0$, we find that the maximum force is, $X_{1max} = -\frac{3}{4}\frac{L_{12}X_2}{L_{11}}$ we get

$$E = \frac{18T}{16}\frac{L_{12}^2X_2^2}{L_{11}} - TL_{22}X_2^2 \quad (13)$$

Coupling parameter is defined by $q = \dfrac{L_{12}^2}{L_{11}L_{22}}$ (13) yields $E = TL_{22}X_2^2\left(\frac{18q^2}{16}-1\right)$ (14)

using $|q| = 1$ in Eq. (14) , and fluxes and force, as well as $L_{22} = \dfrac{T^3\Delta S^2}{(A_{irr}+C_{irr}/\alpha)(\alpha+1)}$

Izumida derived the maximal ecological function as $E_{max} = \dfrac{\Delta T^2\Delta S^2}{32A_{irr}}$ (15)

$$\eta_{E_{max}} = \frac{\Delta T}{4T}\frac{3q^2}{(4-3q^2)} = \frac{3q^2\eta_C}{4(4-3q^2)} \qquad |q| = 1 \qquad \eta_{E_{max}} = \frac{3\Delta T}{4T} = \frac{3}{4}\eta_C + O(\eta_C{}^2) \quad (16)$$

## The Brownian Carnot cycle under eficiente power stochastic model

This regime was studied by Yilmaz et al [4], which consist in the multiplication of the power by the cycle efficiency. The criterion was successfully applied to the Carnot, Brayton, and Diesel cycles, among systems.

From above, the aproach called maximum efficiency power in the context of termal engines is defined as,

$P_E = \eta P$ (17). Now using (5) and (6), plus efficency definition in (16), the efficient power yields

$$P_E = \frac{\left(-W_1^{irr} - W_3^{irr} - \Delta T\Delta S\right)^2}{\left(T_H\Delta S - W_1^{irr}\right)(t_1+t_3)} \quad (18) \qquad W_1^{irr} = A_{irr}/t_1 \qquad W_3^{irr} = A_{irr}/t_3 \qquad P_E(t_1,t_3) \quad \frac{\partial P_E}{\partial t_1} = 0 \quad \frac{\partial P_E}{\partial t_3} = 0$$

$$P_{E_{max}} = \frac{\left(\frac{-A_{irr}}{\gamma} - C_{irr} + \Delta T\Delta S t_3\right)^2}{t_3^2\left(T\Delta S t_3 - \frac{A_{irr}}{\gamma}\right)(\gamma-1)} \qquad \eta_{P_{E_{max}}} = \frac{\beta_1}{2-\alpha_1\eta_C} \qquad \eta_{P_{E_{max}}} = \frac{4\frac{\Delta T}{T}}{2-\frac{4}{3}\frac{\Delta T}{T}} = \frac{2}{3}\eta_C + O(\eta_C{}^2)$$

## The Brownian Carnot cycle under the efficient power criterion using Linear Irreversible Thermodynamics

Taking into account the expression for the power output and efficiency obtained $P_E = \dfrac{T^2(L_{11}X_1 + L_{12}X_2)^2X_1^2}{(L_{21}X_1 + L_{22}X_2)}$

$$\frac{\partial P_E}{\partial X_1} = 0 \Rightarrow X_{1max} = -\frac{r}{6}\frac{L_{22}X_2}{L_{12}} \qquad P_E = \frac{T^2\left[rL_{11}\left(\frac{L_{22}X_2}{6L_{12}}\right) + L_{12}X_2\right]^2\left(\frac{rL_{22}X_2}{6L_{12}}\right)^2}{L_{22}X_2\left(\frac{r}{6}+1\right)}$$

$$r = -q^2 - 4\sqrt{q^4 - 16q^2 + 16}$$

$$q = \frac{L_{12}^2}{L_{11}L_{22}} \Rightarrow P_E = \frac{T^2X_2^3L_{22}^2\left(\frac{r}{6q^2+1}\right)^2\left(\frac{r^2}{36}\right)}{\left(\frac{r}{6}+1\right)} \qquad |q| = 1 \qquad P_{E_{max}} = \frac{\Delta T^2\Delta S^2}{27A}\frac{\Delta T}{T}$$

so

$$\eta_{P_{E_{max}}} = -\frac{\left[\frac{r}{6}\left(\frac{1}{q^2}+1\right)\right]\frac{\Delta T}{T}}{1+\frac{6}{r}} \Rightarrow$$

r = - 4

$|q| = 1$

$$\eta_{P_{E_{max}}} = \frac{2\Delta T}{3T} = \frac{2}{3}\eta_C + + O(\eta_C{}^2)$$

## Results

| | Ecological regime | Efficient power regime | Maximum power regime |
|---|---|---|---|
| Stochastic model | $\eta_{E_{max}} = \frac{\beta\eta_C}{2-\alpha\eta_C}$ | $\eta_{P_{E_{max}}} = \frac{\beta_1}{2-\alpha_1\eta_C}$ | $\eta_{P_{max}} = \frac{\eta_c}{2-\alpha\eta_c}$ |
| | $\eta_{E_{max}} = \frac{3}{4}\eta_C + O(\eta_C^2)$ | $\eta_{P_{E_{max}}} = \frac{2}{3}\eta_C + O(\eta_C{}^2)$ | |
| LIT model | $\eta_{E_{max}} = \frac{3q^2\eta_C}{4(4-3q^2)}$ | $\eta_{P_{E_{max}}} = -\frac{\left[\frac{r}{6}\left(\frac{1}{q^2}+1\right)\right]\eta_C}{1+\frac{6}{r}}$ | $\eta_{P_{max}} = \eta_C\frac{q^2}{(2-q^2)}$ |
| | $\eta_{E_{max}} = \frac{3}{4}\eta_C + O(\eta_C^2)$ | $\eta_{P_{E_{max}}} = \frac{2}{3}\eta_C + O(\eta_C{}^2)$ | $\eta_{P_{max}} = \frac{\eta_C}{2} + O(\eta_C{}^2)$ |

**Table 1 Main results for stochastic model and LIT model for the ecological function regime, the efficient power regime and the máximum power regime.**

## Conclusions

The main results obtained in this work, are shown in Table I, and we remark that, in the case of the maximum efficiency for the ecological function regime, we obtained the following relation (Eq. (14)), which is similar to the main result obtained by Schmiedl et al [7]. (which they called it a ''universal law''). However, we think that in our case this relation is more general because, although the structure is almost the same, the regime in which we work depends on the value of the α and β parameters. For instance, if β = 1, α = 2, we recover the efficiency obtained by Schmiedl et al. [7] for the efficiency at maximum power. But, if we change the value of these parameters we can move to another working regime. It is worthwhile to notice that when we take $t_1 = t_3$ and Airr = Cirr we recover the traditional results for the efficiency at the ecological regime obtained in related papers [10,11].

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