Analysis of Generalized Gibbs States

Talk at the Conference

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Abstract

An exponential family is a manifold of generalized Gibbs state of the form $exp(H)/Tr(exp(H))$, where H belongs to a vector space of (possibly non-commutative) hermitian matrices. Generalized Gibbs states are important in small-scale thermodynamics, they represent equilibrium states regarding several conserved quantities that admit novel operations without heat dissipation [1]. Quantum information theory and condensed matter physics consider a space of local Hamiltonians acting on spins. The entropy distance from this exponential family is a measure of many-body complexity [2,3,4].

This talk is concerned with the geometry and topology of an exponential family and its entropy distance [5]. The maximum-entropy inference map parametrizes the exponential family. This map is continuous in the interior of its domain, the joint numerical range [6]. We describe the points of discontinuity in terms of open mapping theorems and eigenvalue crossings. Because of the discontinuity, the inference map and the entropy distance cannot be approximated through interior points. Instead, it is necessary to study faces (flat portions on the boundary) of the joint numerical range. With local Hamiltonians, this requires studying the faces of the set of quantum marginals.

Abstract

[1] M. Lostaglio, D. Jennings, and T. Rudolph, New Journal of Physics **19**:4, 043008 (2017).

[2] N. Ay, E. Olbrich, N. Bertschinger, and J. Jost, Chaos **21**:3, 037103 (2011).

[3] S. Niekamp, T. Galla, M. Kleinmann, and O. Gühne, J. Phys. A-Math. Gen. **46**:12, 125301 (2013).

[4] B. Zeng, X. Chen, D.-L. Zhou, and X.-G. Wen, Quantum Information Meets Quantum Matter, New York: Springer, 2019.

[5] S. Weis, Journal of Convex Analysis **19**:2, 339–399 (2014).

[6] L. Rodman, I. M. Spitkovsky, A. Szkoła, and S. Weis, Journal of Mathematical Physics **57**:1, 015204 (2016).

Landauer Erasure

According to Landauer, the erasure of one bit of information has an energy cost of $kT \log(2)$, where $k \approx 1.38 \cdot 10^{-23} \mathrm{J/K}$ is the Boltzmann constant and T is the absolute temperature of a heat bath coupled to the memory.

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Theorem (D. Reeb & M. M. Wolf; M. Lostaglio et al. [1])

The states on the combined Hilbert space $\mathcal{H}_M \otimes \mathcal{H}_R$ of the memory and the reservoir before and after the erasure (unitary evolution) satisfy

$$\beta\,\Delta H \geq -\Delta S,$$

where $\beta = 1/(kT)$ is the inverse temperature, H is the energy observable of the heat bath, initially in the state $\rho = e^{-\beta H}/\operatorname{Tr} e^{-\beta H}$, and $S(\rho) = \operatorname{Tr}\rho\log(\rho)$ is the von Neumann entropy of the memory state.

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$$\textstyle\sum_{i\geq 1} \mu_i \, \Delta C_i \geq -\Delta S,$$

where the C_i are conserved observables, possibly noncommutative, of the reservoir, which is initially in the state $\rho = e^{-\sum_{i\geq 1}\mu_i C_i} / \operatorname{Tr} e^{-\sum_{i\geq 1}\mu_i C_i}$, and $S(\rho) = \operatorname{Tr}\rho\log(\rho)$ is the von Neumann entropy of the memory state.

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• Tradeoff between different conserved quantities! Thermal operations without heat dissipation.

• Motivation to study the manifold of generalized Gibbs states $\{e^{-\sum_{i\geq 1}\mu_i C_i} / \operatorname{Tr} e^{-\sum_{i\geq 1}\mu_i C_i} : \mu_i \in \mathbb{R}\}$.

Entropy Distance and Local Hamiltonians

Exponential Families

Let $\mathcal{U} \subseteq M_n$ be a vector space of hermitian matrices. The exponential family associated with $\mathcal{U}$ is $\mathcal{E} = \mathcal{E}(\mathcal{U}) = \{e^A / \operatorname{Tr} e^A : A \in \mathcal{U}\}$.

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The entropy distance from $\mathcal{E}$ is known as a measure of complexity (N. Ay, Annals of Probability **30**:1, 416 (2002)), especially in the following setting [2,3,4].

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Local Hamiltonians

A k-local Hamiltonian is a sum of terms $A_1 \otimes \cdots \otimes A_N \in M_n^{\otimes N}$, each term having at most k non-scalar factors A_i [4]. We denote the space of k-local Hamiltonians by $\mathcal{U}_k$, $\mathcal{E}_k = \mathcal{E}(\mathcal{U}_k)$, and $d_k(\rho) = d_{\mathcal{E}_k}(\rho)$ for states $\rho \in \mathcal{D}$.

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Theorem (S.W. [5])

For all $A \in \mathcal{D} + \mathcal{U}^\perp$ there is a unique state $\pi_{\mathcal{E}}(A) \in (A + \mathcal{U}^\perp) \cap \mathrm{cl}(\mathcal{E})$. For all $\rho \in \mathcal{D}$ and $\tau \in \mathrm{cl}(\mathcal{E})$ we have

a) $S(\rho, \tau) = S(\rho, \pi_{\mathcal{E}}(\rho)) + S(\pi_{\mathcal{E}}(\rho), \tau)$, (Pythagorean theorem)

b) $d_{\mathcal{E}}(\rho) = d_{\mathrm{cl}(\mathcal{E})}(\rho) = S(\rho, \pi_{\mathcal{E}}(\rho))$. (projection theorem)

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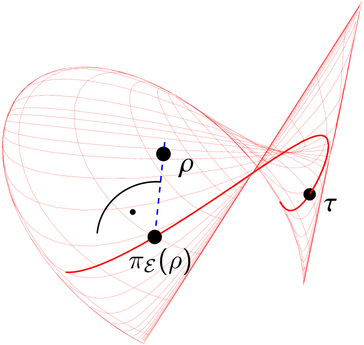
- The map $\Psi = \pi_{\mathcal{E}}|_{\pi_{\mathcal{U}}(\mathcal{D})}$ is the maximum-entropy inference map $\pi_{\mathcal{U}}(\mathcal{D}) \to \mathcal{D}$, which image is $\Psi(\pi_{\mathcal{U}}(\mathcal{D})) = \mathrm{cl}(\mathcal{E})$.

Pictures — Pythagorean Theorem — Joint Numerical Range



Example: 3×3 matrices (S.W. and A. Knauf, Journal of Mathematical Physics **53**:10, 102206 (2012))

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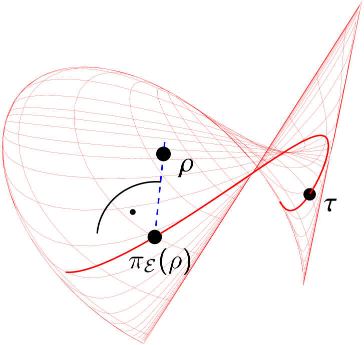

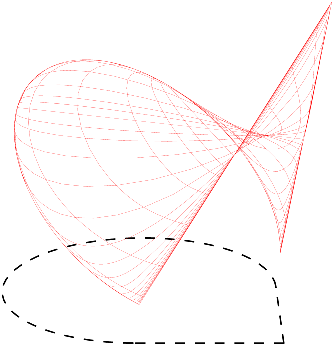

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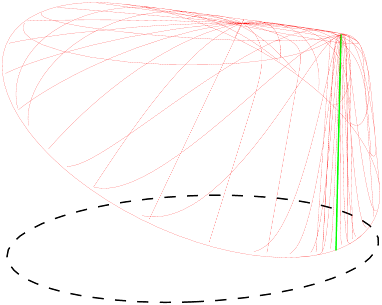
Pictures — Problem Discontinuity of the Inference Ψ



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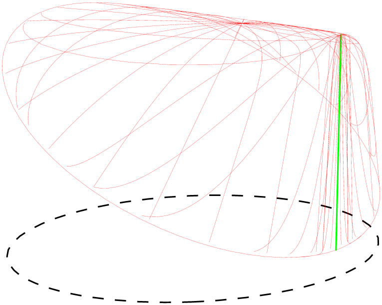

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Note: If the matrices in $\mathcal{U}$ commute, then Ψ is continuous (O. Barndorff-Nielsen, John Wiley & Sons, Chichester, 2014; N. Sukumar and R. J.-B. Wets, SIAM J. Optim. **18**:3, 914–925 (2007)).

Continuity Conditions

Theorem (S.W., CMP 330:3, 1263 (2014))

For each $A \in \pi_{\mathcal{U}}(\mathcal{D})$ the inference map Ψ is continuous at A if and only if the linear map $\pi_{\mathcal{U}}|_{\mathcal{D}}$ is open at $\Psi(A)$.

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Theorem (I. M. Spitkovsky and S.W., JMP 59:12, 121901 (2018))

Let $\mathcal{U} = \mathrm{span}(C_1, C_2)$ and $\lambda(x_1, x_2)$ the smallest eigenvalue of $x_1 C_2 + x_2 C_2$.

- $\nabla\lambda : S^1 \to \partial\pi_{\mathcal{U}}(\mathcal{D})$ parametrizes the boundary of the numerical range,
- one-to-one correspondence of C^1-crossings of $\varphi \mapsto \lambda(e^{\mathrm{i}\,\varphi})$ with the curve of a larger eigenvalue and discontinuities of Ψ at $\nabla\lambda(e^{\mathrm{i}\,\varphi})$,
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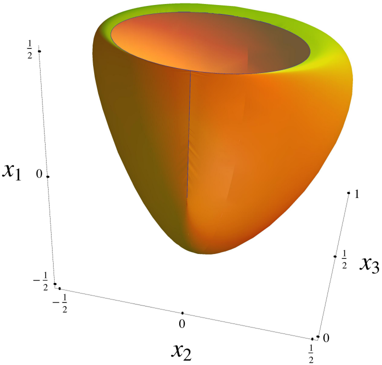
Theorem (L. Rodman et al. [6])

If Ψ is continuous at $A \in \pi_{\mathcal{U}}(\mathcal{D})$, then the dimension of the face-function of $\pi_{\mathcal{U}}(\mathcal{D})$ is lower semi-continuous at A.

Picture — The Dimension of the Face-Function Jumps Up!



The picture shows the set $\{(x_1, x_2, x_3) : x_i = \operatorname{Tr} \rho C_i, \rho \in \mathcal{D}\}$, where

$$F_1 = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$F_2 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$F_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Example by M.-T. Chien and H. Nakazato,

Lin. Alg. Appl. **432**:1, 173 (2010);

K. Szymański, S.W., K. Życzkowski,

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Algorithms

Question

Develop algorithms that compute the values of the
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and that tell us how far a point is from a discontinuity.

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Regarding the discontinuity, we need to take into account the faces of the joint numerical range $\pi_{\mathcal{U}}(\mathcal{D})$. Concerning local Hamiltonians, $\pi_{\mathcal{U}_k}(\mathcal{D})$ is the convex set of k-party marginals. Even for two-party marginals of three qubits we know quite little about the set of marginals:

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• maximal faces can be efficiently sampled from the extreme points of the dual spectrahedron, and can be tested algebraically (S.W. and J. Gouveia, `arXiv:2103.08360`).

Thank you for your attention!

These slides were created with LaTeX (beamer class and bclogo-package). The graphics were drawn with `Wolfram Mathematica`.